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*First time in print.
Boundary Value Problems of Mathematical Physics

Volume II

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To Alice
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Table of Contents

5. DISTRIBUTIONS AND GENERALIZED SOLUTIONS
   5.1 Introduction .................................................. 1
   5.2 Test Functions ................................................. 3
   5.3 Distributions .................................................. 4
   5.4 Convergence of Distributions .............................. 10
   5.5 Additional Properties of Distributions ............... 17
   5.6 Fourier Transforms .......................................... 23
   5.7 Partial Differential Equations for Distributions .... 39
   5.8 Fundamental Solutions ...................................... 48
   5.9 Classification of Partial Differential Equations .... 72

6. POTENTIAL THEORY
   6.1 Introduction .................................................. 88
   6.2 Interior Dirichlet Problem for the Unit Circle ....... 90
   6.3 Some Properties of Harmonic Functions ................. 99
   6.4 Surface Layers ............................................... 110
   6.5 Integral Equations of Potential Theory ............... 122
   6.6 Green's Function for the Negative Laplacian .......... 130
   6.7 Methods for Determining the Green's Function ....... 146
   6.8 Some Physical Applications of Potential Theory .... 171

7. EQUATIONS OF EVOLUTION
   7.1 Introduction .................................................. 194
   7.2 Causal Green's Function for Heat Conduction ......... 197
   7.3 Methods for Finding the Causal Green's Function .... 204
   7.4 Uniqueness and Continuous Dependence on the Data ... 222
   7.5 Miscellaneous Topics Related to the Heat Equation ... 227
   7.6 Preliminary Considerations for the Undamped Wave Equation ........................................... 243
   7.7 Causal Green's Function for the Wave Equation ....... 246
CONTENTS

7.8 Problems in One Space Dimension 249
7.9 Problems in More than One Dimension 253
7.10 Wave Equation with External Damping 257
7.11 Monochromatic Excitation and the Principle of Limiting Absorption 259
7.12 Green's Function for the Helmholtz Operator and Applications 265
7.13 Half-Plane Excited by a Line Source or a Plane Wave 281
7.14 Representation of Solutions of the Helmholtz Equation in Exterior Domains 294
7.15 Scattering Problem 299
7.16 Wiener-Hopf Method 311

8. VARIATIONAL AND RELATED METHODS

8.1 Introduction 332
8.2 Best Approximation in a Subspace 335
8.3 Maximum Theorem 337
8.4 Ritz-Rayleigh Method 340
8.5 Complementary Variational Principles 344
8.6 Capacity Problem 350
8.7 Natural Boundary Conditions 352
8.8 Indefinite and Nonsymmetric Operators 355
8.9 Other Methods for Upper Bounds to Functionals Associated with Positive Operators 358
8.10 Method of Least Squares 361
8.11 Extremal Principles for Eigenvalue Problems on Euclidean n Space 369
8.12 Eigenvalue Problems in Hilbert Space 372
8.13 Lower Bounds to Eigenvalues 381

APPENDIX A. SPHERICAL HARMONICS 393

APPENDIX B. ASYMPTOTIC EXPANSIONS 399

SUGGESTED ADDITIONAL READINGS 403

INDEX 405
Preface to the Classics Edition

I am pleased and honored that SIAM is reissuing the two volumes of *Boundary Value Problems of Mathematical Physics*, originally published (1967–68) in Macmillan’s Series in Advanced Mathematics and Theoretical Physics under the editorship of Mark Kac, C. C. Lin, and George Uhlenbeck. Soon afterwards, Macmillan was bought by Crowell-Collier, who discontinued the series and the marketing of graduate-level books. Thus, despite widespread university adoptions, the two volumes went out of print in 1974, with the publishing rights reverting to me.

As my research interests were changing, I decided to write a new book, *Green’s Functions and Boundary Value Problems*, which was published by Wiley-Interscience in 1979, with a second edition appearing in late 1997. This book covers some of the topics of the Macmillan volumes, but somewhat more abstractly and with greater emphasis on nonlinear problems.

In light of the many developments in computational applied mathematics since 1968, the prospective reader may well ask whether the present reissue is no more than an exercise in nostalgia! To allay such understandable doubts, let me point out that the Macmillan volumes are still used—through special arrangements—at a number of universities. They report that one of the attractive features of these volumes is the large number of concrete examples of boundary value problems for partial differential equations. For instance, there is a substantial treatment of the Helmholtz equation and scattering theory, subjects which play a central role in contemporary inverse problems in acoustics and electromagnetic theory. More generally, an examination of the table of contents of the original volumes confirms that the topics covered do belong in the arsenal of today’s engineer, physical scientist, and applied mathematician. It must be acknowledged, however, that the practical needs of the modern workplace demand supplementary credentials, particularly in the area of numerical and computational methods.

In conclusion, let me thank SIAM and especially Robert E. O’Malley, Jr. for undertaking this reissue. Although I have received encouragement from many sources over the years, I particularly want to thank my long-time friend, W. Edward Olmstead, for his continuing interest in these volumes.

I. S.
The first volume of *Boundary Value Problems of Mathematical Physics*, published in 1967, develops the mathematical foundations required for the study of linear partial differential equations, the subject matter of the present volume.

The field of partial differential equations has grown to such an extent in recent years that it would be impossible to cope adequately with all its aspects in a book of manageable size. Since I am addressing myself principally to graduate students in engineering and the physical sciences, I have emphasized methods for finding solutions in useful form. Even so, the more abstract questions of uniqueness, existence, and stability are not entirely neglected because of the qualitative insight they often provide.

The starting point chosen here for the study of partial differential equations is the notion of a fundamental solution, that is, the response of the physical system to a concentrated or impulsive forcing function. The mathematical formulation requires a discussion of the theory of distributions and generalized solutions of differential equations. In connection with boundary value problems, the Green’s function, a fundamental solution satisfying specific boundary conditions, plays a central role. Much effort is devoted to methods for constructing the Green’s function. The principal approaches use eigenfunction expansions, transform methods, and integral equations; the interplay among these techniques is stressed and often a problem is solved in alternative ways.

These methods are mainly applied to Laplace’s equation, the diffusion equation, and the wave equation, but the theory is formulated in terms broad enough for more general use.

Special features include the treatment of scattering theory, Wiener-Hopf equations, and variational principles.

In conclusion I wish to thank my students and colleagues for their encouragement and Mrs. J. Prangley for her skillful typing of the manuscript.

I. S.
Chapter 5
DISTRIBUTIONS
AND GENERALIZED SOLUTIONS

5.1 INTRODUCTION

Modern developments in partial differential equations require a thorough grounding in the theory of distributions in more than one variable. The one-dimensional theory of distributions was presented in Section 1.3,† and the ideas which motivated that discussion will again serve as a guide. The principal novelty will be the greater variety of distributions confronting us; for instance, in 3-space we can have sources concentrated not only at points but also on curves and surfaces, thereby giving rise to equivalent volume source densities which are clearly singular in varying degrees. Before proceeding with the theory of distributions, we introduce some useful notation and a few definitions.

Euclidean n-space is denoted by \( R_n \) and a point in \( R_n \) is labeled \( x = (x_1, \ldots, x_n) \), where \( x_1, \ldots, x_n \) are Cartesian coordinates with respect to a fixed frame of reference. We shall write

\[
r = |x| = (x_1^2 + \cdots + x_n^2)^{1/2}
\]

for the distance between the point \( x \) and the origin. An element of volume \( dx_1 \cdots dx_n \) will be abbreviated \( dx \), so that the integral of a function \( f \) of position over a region \( R \) is written in either of the forms

\[
\int_R f(x_1, \ldots, x_n) dx_1 \cdots dx_n, \quad \int_R f(x) dx.
\]

We shall also consider integrals over hypersurfaces of dimension \( n - 1 \) in \( R_n \). Thus in \( R_3 \) a hypersurface is an ordinary surface, whereas in \( R_2 \) it is an

† Chapters 1 through 4 constitute Volume I.
ordinary curve. Typically a hypersurface is denoted by the letter \( \sigma \) and an element of surface on \( \sigma \) by \( dS \). Occasionally a subscript on \( dS \) will be used to indicate the variable being integrated; for instance, if \( f(x, \xi) \) is a function depending on two points in \( R_\eta \) and we are integrating \( f \) on \( \sigma \) with respect to the variable \( x \), we write

\[
\int_\sigma f(x, \xi) dS_x.
\]

**Definition.** A function \( f(x) \) is said to be *locally integrable* in \( R_\eta \) if \( \int_R |f| dx \) is finite for every bounded region \( R \) in \( R_\eta \).

Not only are all continuous and piecewise continuous functions locally integrable, but also functions having infinite singularities which are not too severe. For instance, the function \( 1/|x|^k \) is locally integrable in \( R_\eta \) if \( k < n \), as is easily verified by passing to spherical coordinates.

**Definition.** A function \( f(x) \) is said to be *locally integrable on the hypersurface \( \sigma \) if \( \int_\sigma |f| dS \) is finite for every bounded subset \( \sigma' \) of \( \sigma \).

We observe that if \( \sigma \) is a hypersurface in \( R_\eta \), the singularities of \( f(x) \) will have to be somewhat milder to ensure local integrability on \( \sigma \). In fact, if \( \xi \) is a point on \( \sigma \), \( 1/|x - \xi|^k \) is locally integrable on \( \sigma \) only if \( k < n - 1 \).

**Definition.** The *support* of \( f(x) \) is the closure of the set of points on which \( f(x) \neq 0 \).

As an example consider \( f(x) = \sin x \), where \( x \) is a real variable in \( R_1 \). Then the support of \( f(x) \) consists of the whole real line even though \( f(x) \) vanishes at \( x = n\pi \).

Next we introduce a concise notation for partial derivatives and differential operators in \( n \) independent variables. Let \( k \) be an \( n \)-dimensional "vector" whose components are nonnegative integers; we shall call \( k \) a *multiindex* of order \( n \). For any multiindex \( k = (k_1, \ldots, k_n) \), we define

\[
|k| = k_1 + \cdots + k_n,
\]

\[
D^k = \frac{\partial^{|k|}}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}} = \frac{\partial^{k_1+\cdots+k_n}}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}},
\]

with the understanding that, if any component of \( k \) is zero, the differentiation with respect to the corresponding variable is omitted. As an example, if \( n = 3 \) and \( k = (2, 0, 5) \), then

\[
D^k = \frac{\partial^7}{\partial x_1^2 \partial x_3^5}.
\]
An arbitrary linear differential operator \( L \) of order \( p \) in \( n \) variables can be written
\[
L = \sum_{|k| \leq p} a_k(x)D^k,
\]
(5.1)
where \( a_k(x) = a_{k_1, \ldots, k_n}(x_1, \ldots, x_n) \) are arbitrary functions and the sum in \( L \) is taken over all multiindices \( k \) of order \( n \). For instance, the most general linear differential operator of order 2 in two variables is
\[
L = \sum_{|k| \leq 2} a_k(x)D^k = a_{2,0} \frac{\partial^2}{\partial x_1^2} + a_{1,1} \frac{\partial^2}{\partial x_1 \partial x_2} + a_{0,2} \frac{\partial^2}{\partial x_2^2} + a_{1,0} \frac{\partial}{\partial x_1} + a_{0,1} \frac{\partial}{\partial x_2} + a_{0,0}.
\]

5.2 TEST FUNCTIONS

**Definition.** A function \( \varphi(x) = \varphi(x_1, \ldots, x_n) \) will be called a test function (or sometimes an \( n \)-dimensional test function) if it satisfies the two criteria:

1. It is infinitely differentiable throughout \( R_n \). Thus, if \( k \) is a multiindex of order \( n \), \( D^k \varphi \) exists.
2. There exists \( A \) such that \( \varphi(x) = 0 \) for \( |x| > A \).

The space of all \( n \)-dimensional test functions is denoted by \( K_n \).†

The following is an example of a test function:
\[
\varphi(x) = \begin{cases} 
0, & r \geq a; \\
\exp\left(\frac{1}{r^2 - a^2}\right), & r < a.
\end{cases}
\]
(5.2)

The following properties of test functions are easily proved.

1. The space \( K_n \) is a linear space. Thus, if \( \varphi_1 \) and \( \varphi_2 \) are in \( K_n \) and \( \alpha_1 \) and \( \alpha_2 \) are constants, \( \alpha_1 \varphi_1 + \alpha_2 \varphi_2 \) is also in \( K_n \).
2. If \( \varphi \) is in \( K_n \), so is \( D^k \varphi \).
3. If \( \varphi \) is in \( K_n \) and \( f(x_1, \ldots, x_n) \) is infinitely differentiable, then \( f\varphi \) is in \( K_n \).
4. If \( \varphi(x_1, \ldots, x_m) \) is an \( m \)-dimensional test function and \( \psi(x_{m+1}, \ldots, x_n) \) is an \( (n-m) \)-dimensional test function, then \( \varphi\psi \) is an \( n \)-dimensional test function in the variables \( x_1, \ldots, x_n \).

**Convergence in the Space of Test Functions**

A sequence of test functions \( \{\varphi_m(x)\} \) is a *null sequence in \( K_n \) if and only if*

1. All \( \varphi_m(x) \) vanish outside a common finite sphere.
2. \( \varphi_m(x) \) and all its partial derivatives approach 0 uniformly over \( R_n \) as \( m \) approaches infinity. Thus, for each multiindex \( k \) of order \( n \),
\[
\lim_{m \to \infty} D^k \varphi_m = 0 \text{ uniformly over } R_n.
\]
† The space \( K_1 \) was denoted by \( D \) in volume I.
5.3 DISTRIBUTIONS

We say that \( t \) is a functional on \( K_n \) if there exists a rule which assigns to each \( \varphi \) in \( K_n \) a real number \( \langle t, \varphi \rangle \). The functional \( t \) is said to be linear if, whenever \( \alpha_1 \) and \( \alpha_2 \) are real numbers and \( \varphi_1 \) and \( \varphi_2 \) are in \( K_n \), we have

\[
\langle t, \alpha_1 \varphi_1 + \alpha_2 \varphi_2 \rangle = \alpha_1 \langle t, \varphi_1 \rangle + \alpha_2 \langle t, \varphi_2 \rangle.
\]

A linear functional \( t \) is continuous if, whenever \( \{\varphi_m(x)\} \) is a null sequence in \( K_n \), the sequence of real numbers \( \{\langle t, \varphi_m \rangle\} \) converges to zero as \( m \) tends to infinity.

A continuous linear functional on \( K_n \) is known as a distribution (or \( n \)-dimensional distribution). If \( t \) is a distribution, the number \( \langle t, \varphi \rangle \) is sometimes called the action of \( t \) on \( \varphi \).

**Theorem 1.** Every locally integrable function \( f(x) \) defines a distribution through the formula

\[
\langle f, \varphi \rangle = \int_{\mathbb{R}^n} f(x) \varphi(x) dx = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \ldots, x_n) \varphi(x_1, \ldots, x_n) dx_1 \cdots dx_n.
\]

**Proof.** It is clear that a linear functional on \( K_n \) has been defined. To prove continuity, let \( \{\varphi_m(x)\} \) be a null sequence all of whose elements vanish outside the finite sphere \( R \). Then

\[
|\langle f, \varphi_m \rangle| \leq \left[ \max_{x \in R} |\varphi_m(x)| \right] \int_R |f| dx.
\]

Since \( \{\varphi_m\} \) is a null sequence, it follows that \( \lim_{m \to \infty} \max_{x \in R} |\varphi_m(x)| = 0 \). The local integrability of \( f \) guarantees that \( \int_R |f| dx \) is finite, so that \( \lim_{m \to \infty} \langle f, \varphi_m \rangle = 0 \) and the functional is continuous.

**Definition.** A distribution is regular if it can be written in the form (5.3) with \( f(x) \) a locally integrable function. All other distributions are singular. Even in this latter case we sometimes use formula (5.3) symbolically. Thus each distribution \( t \) is assigned a symbolic or generalized function \( t(x) \). The symbols \( t, \langle t, \varphi \rangle \), and \( t(x) \) are used interchangeably to describe a distribution.

**Remark.** If \( f_1(x) \) and \( f_2(x) \) are locally integrable functions which are equal almost everywhere (that is, such that \( \int_{\mathbb{R}^n} |f_1 - f_2| \ dx = 0 \)), then \( \langle f_1, \varphi \rangle = \langle f_2, \varphi \rangle \) for each test function \( \varphi \) in \( K_n \). Thus \( f_1 \) and \( f_2 \) define the same distribution and are therefore considered equivalent functions within the present theory. On the other hand, two different regular distributions stem from different functions (that is, functions which are not equal almost everywhere).
If \( t_1 \) and \( t_2 \) are distributions on \( K_n \), \( t_1 + t_2 \) is the distribution on \( K_n \) defined by
\[
\langle t_1 + t_2, \varphi \rangle = \langle t_1, \varphi \rangle + \langle t_2, \varphi \rangle.
\]
If \( \alpha \) is a constant and \( t \) is a distribution on \( K_n \), \( \alpha t \) is the distribution on \( K_n \) defined by
\[
\langle \alpha t, \varphi \rangle = \alpha \langle t, \varphi \rangle.
\]
The zero distribution \( 0 \) on \( K_n \) has the property
\[
\langle 0, \varphi \rangle = 0
\]
for every test function \( \varphi \).

It is clear that with these definitions the set of all distributions on \( K_n \) is a linear space.

**EXAMPLES**

**Example 1.** Let \( R \) be a fixed region. Consider the functional \( \langle H_R, \varphi \rangle = \int_R \varphi(x)dx \). This functional is clearly linear and can be written in the form \( \int_{R^n} H_R(x)\varphi(x)dx \), where \( H_R(x) \) is the locally integrable function which is 1 in \( R \) and 0 outside \( R \). The functional \( \langle H_R, \varphi \rangle \) is therefore a regular distribution.

**Example 2.** Let \( \xi \) be a fixed point. Consider the functional \( \langle \delta_\xi, \varphi \rangle = \varphi(\xi) \) which thus assigns to each test function \( \varphi \) its value at the point \( \xi \). This functional is clearly linear; moreover, if \( \{\varphi_m(x)\} \) is a null sequence of test functions, then the sequence of numbers \( \varphi_m(\xi) \) approaches 0 as \( m \to \infty \), so that \( \delta_\xi \) is continuous and hence a distribution. By the same argument used in Section 1.3, it can easily be shown that \( \delta_\xi(x) \) is a singular distribution. This distribution is known as the \( n \)-dimensional Dirac distribution and we write
\[
\langle \delta_\xi, \varphi \rangle = \varphi(\xi) = \int_{R^n} \delta_\xi(x)\varphi(x)dx,
\]
where the last equality defines symbolically the generalized function \( \delta_\xi(x) \).

We visualize \( \delta_\xi(x) \) as the (highly singular) volume source density for a unit source concentrated at \( \xi \). The distribution \( \delta_0 \) will be denoted by \( \delta \).

**Example 3.** *Translation of a distribution.* If \( f(x) \) is locally integrable, the \( a \) translate of \( f(x) \) is the locally integrable function \( f(x - a) \). We have
\[
\langle f(x - a), \varphi(x) \rangle = \int_{R^n} f(x - a)\varphi(x)dx = \int_{R^n} f(x)\varphi(x + a)dx
\]
\[
= \langle f(x), \varphi(x + a) \rangle.
\]
We are therefore led to define the \( a \) translate \( t(x - a) \) of an arbitrary distribution \( t(x) \) by the formula
\[
\langle t(x - a), \varphi(x) \rangle = \langle t(x), \varphi(x + a) \rangle. \tag{5.4}
\]
It is quite clear that \( t(x - a) \) is a distribution. We observe that

\[
\langle \delta(x - \xi), \varphi(x) \rangle = \langle \delta(x), \varphi(x + \xi) \rangle = \varphi(\xi),
\]

so that

\[
\delta(x - \xi) = \delta_\xi(x).
\]

In the remainder of our work we shall use the notation \( \delta(x - \xi) \) instead of \( \delta_\xi(x) \).

**Example 4.** Let \( l \) be a unit vector defining a direction in space in \( R_n \). A unit dipole at \( \xi \) with axis \( l \) is the limit as \( \varepsilon \to 0 \) of the source configuration consisting of a concentrated source \( 1/\varepsilon \) at \( \xi + (\varepsilon/2)l \) and a concentrated source \( -1/\varepsilon \) at \( \xi - (\varepsilon/2)l \). To interpret this dipole as functional, we note that its action on a test function is given by

\[
\lim_{\varepsilon \to 0} \left[ \frac{1}{\varepsilon} \varphi \left( \xi + \frac{\varepsilon}{2} l \right) - \frac{1}{\varepsilon} \varphi \left( \xi - \frac{\varepsilon}{2} l \right) \right] = \frac{d\varphi}{dl}(\xi).
\]

This functional is seen to be linear and continuous. Thus we have defined a distribution known as a *dipole distribution*.

**Example 5.** If \( \alpha \) is a nonzero constant and \( f(x) \) is a locally integrable function, \( f(\alpha x) \) is also locally integrable. The distribution corresponding to \( f(\alpha x) \) is \( \langle f(\alpha x), \varphi(x) \rangle = \int_{R^n} f(\alpha x) \varphi(x) dx \). Setting \( \alpha x = y \) and observing that the limits of integration are reversed if \( \alpha \) is negative, we have

\[
\langle f(\alpha x), \varphi(x) \rangle = \frac{1}{|\alpha|^n} \int_{R^n} f(y) \varphi \left( \frac{y}{\alpha} \right) dy = \frac{1}{|\alpha|^n} \left\langle f(x), \varphi \left( \frac{x}{\alpha} \right) \right\rangle.
\]

For any distribution \( t(x) \) we define \( t(\alpha x) \) as the distribution

\[
\langle t(\alpha x), \varphi(x) \rangle = \frac{1}{|\alpha|^n} \left\langle t(x), \varphi \left( \frac{x}{\alpha} \right) \right\rangle.
\]  

(5.5)

**Example 6.** Let \( \sigma \) be a hypersurface of dimension \( n - 1 \) in \( R_n \) and let \( dS \) stand for an element of surface area on \( \sigma \). In \( R_3 \), \( \sigma \) would be an ordinary surface, whereas in \( R_2 \), \( \sigma \) would be a plane curve.

Consider the following functional on \( K_n \):

\[
\langle t, \varphi \rangle = \int_{\sigma} a(x) \varphi(x) dS,
\]

(5.6)

where \( a(x) \) is a function defined on \( \sigma \) and locally integrable over \( \sigma \). It is easily seen that \( t \) is a distribution which is singular. We can interpret \( t \) in terms of surface sources as follows. Suppose that sources are spread on \( \sigma \) with *surface density* \( a(x) \). The surface element \( dS_\xi \) at \( \xi \) on \( \sigma \) carries a concentrated source
of strength \(a(\xi)dS_\xi\), and therefore the corresponding volume source density is \(a(\xi)dS_\xi \delta(x - \xi)\). Thus the total volume density can be formally expressed as

\[
\int_\sigma a(\xi)\delta(x - \xi)dS_\xi.
\]

Proceeding heuristically we observe that the action of \(\int_\sigma a(\xi)\delta(x - \xi)dS_\xi\) on a test function \(\varphi\) in \(K_n\) is

\[
\left\langle \int_\sigma a(\xi)\delta(x - \xi)dS_\xi, \varphi \right\rangle = \int_{R_n} d\varphi(x) \int_\sigma a(\xi)\delta(x - \xi)dS_\xi.
\]

Calculating the integral on the right by first integrating with respect to \(R_n\), we obtain

\[
\left\langle \int_\sigma a(\xi)\delta(x - \xi)dS_\xi, \varphi \right\rangle = \int_\sigma a(\xi)\varphi(\xi)dS_\xi.
\]

We are therefore entitled to speak of (5.6) as the distribution corresponding to a surface layer of sources spread on \(\sigma\) with surface density \(a(x)\). Such a layer is known as a *simple layer*.

**Product of a Distribution and a Function**

If \(f(x)\) and \(g(x)\) are locally integrable, the product \(f(x)g(x)\) need not be locally integrable (as is shown by the example \(f = g = 1/\sqrt{|x|}\) in one dimension). Even if \(fg\) is locally integrable, the action \(\left\langle fg, \varphi \right\rangle\) may not be related to the individual actions of \(f\) and \(g\). On the other hand, if \(g\) is infinitely differentiable, \(g\varphi\) is a test function and we have

\[
\left\langle fg, \varphi \right\rangle = \int_{R_n} f(x)g(x)\varphi(x)dx = \left\langle f, g\varphi \right\rangle.
\]

Similarly, if \(f(x)\) is infinitely differentiable and \(t(x)\) is an arbitrary distribution, we shall define \(ft\) by

\[
\left\langle ft, \varphi \right\rangle = \left\langle t, f\varphi \right\rangle. \quad (5.7)
\]

There are distributions \(t\) (see Examples 2 and 3 below) for which \(ft\) can be given a distributional definition even though \(f\) is not infinitely differentiable, but such definitions must be made on an ad hoc basis.

**Differentiation of Distributions**

If \(f(x)\) is a locally integrable function whose derivative \(\frac{\partial f}{\partial x_1}\) is locally integrable, the functional corresponding to \(\frac{\partial f}{\partial x_1}\) is

\[
\left\langle \frac{\partial f}{\partial x_1}, \varphi \right\rangle = \int_{R_n} \frac{\partial f}{\partial x_1} \varphi dx.
\]
Integration by parts yields
\[ \left\langle \frac{\partial f}{\partial x_1}, \varphi \right\rangle = - \int_{\mathbb{R}^n} f \frac{\partial \varphi}{\partial x_1} \, dx = - \left\langle f, \frac{\partial \varphi}{\partial x_1} \right\rangle. \]

This last property is used to define the derivative of any distribution \( t \).

**DEFINITION.** Given a distribution \( t \), we define \( \partial t/\partial x_1 \) from
\[ \left\langle \frac{\partial t}{\partial x_1}, \varphi \right\rangle = \left\langle t, - \frac{\partial \varphi}{\partial x_1} \right\rangle = - \left\langle t, \frac{\partial \varphi}{\partial x_1} \right\rangle. \quad (5.8) \]

It is clear that \(- \partial \varphi/\partial x_1\) is a test function, so that we have defined a functional which is obviously linear. Moreover, if \( \{ \varphi_m \} \) is a null sequence so is \( \{- \partial \varphi_m/\partial x_1\} \), so that the continuity of \( t \) implies the continuity of \( \partial t/\partial x_1 \).

Hence we have defined a distribution \( \partial t/\partial x_1 \) from (5.8).

By repeated application of the definition we see that a distribution can be differentiated as often as desired with respect to any combination of the variables \( x_1, \ldots, x_n \):
\[ \langle D^k t, \varphi \rangle = \left\langle \frac{\partial^{k_1}}{\partial x_1^{k_1}} \cdots \frac{\partial^{k_n}}{\partial x_n^{k_n}}, \varphi \right\rangle = (-1)^{k_1} \langle t, D^k \varphi \rangle. \quad (5.9) \]

**EXAMPLES**

**Example 1.**
\[ \left\langle - \frac{\partial}{\partial x_1} \delta(x - \xi), \varphi \right\rangle = \left\langle \delta(x - \xi), \frac{\partial \varphi}{\partial x_1} \right\rangle = \frac{\partial \varphi}{\partial x_1}(\xi). \]

Thus \(- (\partial/\partial x_1) \delta(x - \xi)\) is the generalized function corresponding to a unit dipole at \( \xi \) oriented along the positive \( x_1 \) axis. If the dipole's axis is \( l \), its source density is \(- (d/dx_l) \delta(x - \xi)\), where \( d/dx_l \) denotes differentiation in the \( l \) direction at the point \( x \).

**Example 2.** If \( f(x) \) is infinitely differentiable, we have
\[ \left\langle f(x) \delta(x - \xi), \varphi \right\rangle = \left\langle \delta(x - \xi), f\varphi \right\rangle = f(\xi)\varphi(\xi). \]

The definition
\[ \left\langle f(x) \delta(x - \xi), \varphi \right\rangle = f(\xi)\varphi(\xi) \]

can be used even when \( f(x) \) is only continuous at \( x = \xi \). We observe that
\[ f(x) \delta(x - \xi) = f(\xi) \delta(x - \xi). \quad (5.10) \]

**Example 3.** If \( f(x) \) is infinitely differentiable, we have
\[ \left\langle f(x) \frac{\partial}{\partial x_1} \delta(x - \xi), \varphi \right\rangle = \left\langle \frac{\partial}{\partial x_1} \delta(x - \xi), f\varphi \right\rangle = - \frac{\partial}{\partial x_1} \left( f\varphi \right) \bigg|_{x=\xi} \]
\[ = -f(\xi) \frac{\partial \varphi}{\partial x_1}(\xi) - \frac{\partial f}{\partial x_1}(\xi) \varphi(\xi). \]
This can be used as a definition for \( f(x)(\partial/\partial x_1)\delta(x - \xi) \) even if \( f \) is only continuously differentiable at \( x = \xi \). We have the equation

\[
f(x) \frac{\partial \delta(x - \xi)}{\partial x_1} = f(\xi) \frac{\partial \delta(x - \xi)}{\partial x_1} - \frac{\partial f}{\partial x_1}(\xi)\delta(x - \xi). \tag{5.11}
\]

**Example 4.** Let \( L \) be the general linear differential operator of order \( p \) [see (5.1)], with infinitely differentiable coefficients \( a_k(x) \). If \( t \) is an arbitrary distribution, the derived distribution \( Lt \) is well defined by using (5.7) and (5.9). We have

\[
\langle Lt, \varphi \rangle = \left\langle \sum_{|k| \leq p} a_k D^k t, \varphi \right\rangle = \left\langle t, \sum_{|k| \leq p} (-1)^{|k|} D^k (a_k \varphi) \right\rangle.
\]

The \( p \)th-order operator appearing on the right side of the above is known as the *formal adjoint* of \( L \) and is denoted by \( L^* \). Thus

\[
L^* \varphi = \sum_{|k| \leq p} (-1)^{|k|} D^k (a_k \varphi). \tag{5.12}
\]

If \( L = L^* \), the operator is *formally self-adjoint*. We observe that the operator

\[
\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}
\]

is formally self-adjoint, so that

\[
\langle \nabla^2 t, \varphi \rangle = \langle t, \nabla^2 \varphi \rangle. \tag{5.13}
\]

**Values of a Generalized Function**

One of the purposes in developing the theory of distributions is to circumvent the limitations inherent in point functions. Nevertheless it is useful to ascribe to a generalized function properties similar to those of a point function and, in fact, we have already done this when we introduced such operations as differentiation for a generalized function. On the other hand, it is clearly useless to try to assign a value to a generalized function at a point \( x_0 \) since we know that two locally integrable functions which differ in value at the point \( x_0 \) may generate the same distribution. However, it is possible to define in a meaningful way the notion of the values of a generalized function in an open region \( R \). Let us first consider what should be meant by the statement that the generalized function \( t(x) \) vanishes in the open region \( R \). Thus from a knowledge of \( \langle t, \varphi \rangle \) we want to determine whether or not \( t(x) \) vanishes in \( R \); if \( t \) were a function we could conclude that \( t(x) \) vanishes almost everywhere in \( R \) (that is, \( \int_R |t|dx = 0 \)) if \( \langle t, \varphi \rangle = 0 \) for every test function \( \varphi \) whose support is contained in \( R \). This leads to the

**Definition.** A distribution \( t(x) \) is said to *vanish* in the open region \( R \) if \( \langle t, \varphi \rangle = 0 \) for all the test functions \( \varphi(x) \) whose support is contained in \( R \).
DEFINITION. Two distributions $t_1(x)$ and $t_2(x)$ are said to be equal in $R$ if $t_1 - t_2$ vanishes in $R$.

EXAMPLES

Example 1. Let $R$ be the open region consisting of all of $R_n$ excluding only the origin. A test function $\varphi(x)$ whose support is contained in $R$ must vanish identically in some neighborhood of the origin and therefore

$$\langle \delta, \varphi \rangle = \varphi(0) = 0, \quad \langle D^k \delta, \varphi \rangle = (-1)^{|k|}(D^k \varphi)(0) = 0$$

for any such test function. Hence we can say that $\delta(x)$ and $D^k \delta(x)$ vanish for $x \neq 0$.

Example 2. Consider the distribution $t$ corresponding to a simple layer on $\sigma$. Then $\langle t, \varphi \rangle = \int_\sigma a(x) \varphi(x) dS$, which vanishes if the support of $\varphi$ is contained in the complement of $\sigma$ with respect to $R_n$. Thus $t$ vanishes for $x$ not on $\sigma$.

5.4 CONVERGENCE OF DISTRIBUTIONS

Distributions Depending on a Parameter

Let $t_\alpha(x)$ be a distribution depending parametrically on $\alpha$; that is, for each real number $\alpha$ a distribution $t_\alpha(x)$ is defined. We say that $t_\alpha(x)$ converges as $\alpha \to \alpha_0$ if, for each test function $\varphi(x)$, the set of numbers $\langle t_\alpha, \varphi \rangle$ has a limit as $\alpha$ approaches $\alpha_0$. The limiting value of $\langle t_\alpha, \varphi \rangle$ associated with $\varphi$ is a number which we write $\langle t, \varphi \rangle$. The set of values $\langle t, \varphi \rangle$ is clearly a linear functional on $K_n$. Moreover, it can be shown that $\langle t, \varphi \rangle$ is continuous on $K_n$, so that $t$ is a distribution. We then write

$$\lim_{\alpha \to \alpha_0} t_\alpha(x) = t(x)$$

or

$$\lim_{\alpha \to \alpha_0} \langle t_\alpha, \varphi \rangle = \langle t, \varphi \rangle,$$

and the distribution $t_\alpha$ is said to converge to $t$ as $\alpha$ approaches $\alpha_0$.

As a particular case of the above concept of convergence, consider the case where $\alpha$ runs through the positive integers. We say that the sequence $t_m(x)$ converges to $t(x)$ if, for each $\varphi$,

$$\lim_{m \to \infty} \langle t_m, \varphi \rangle = \langle t, \varphi \rangle.$$

Similarly, if $t_1(x) + t_2(x) + \cdots$ is an infinite series of distributions, we say that $\sum_{k=1}^{\infty} t_k(x)$ converges to $t(x)$ if and only if the sequence of partial sums $s_m(x) = \sum_{k=1}^{m} t_k(x)$ converges to $t(x)$.

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We easily prove the theorem: Every convergent sequence or series of distributions can be differentiated termwise with respect to any combination of the independent variables \(x_1, \ldots, x_n\).

**Proof.** It will be sufficient to show that if \(\lim_{a \to a_0} t_a = t\), then \(\lim_{a \to a_0} \frac{\partial t_a}{\partial x_1} = \frac{\partial t}{\partial x_1}\). In fact, we have

\[
\lim_{a \to a_0} \left\langle \frac{\partial t_a}{\partial x_1}, \varphi \right\rangle = \lim_{a \to a_0} \left\langle t_a, -\frac{\partial \varphi}{\partial x_1} \right\rangle = \left\langle t, -\frac{\partial \varphi}{\partial x_1} \right\rangle = \left\langle \frac{\partial t}{\partial x_1}, \varphi \right\rangle.
\]

**EXAMPLES**

**Example 1. Differentiation with respect to a parameter.** Let \(t_a(x)\) be a distribution depending on the parameter \(a\). With \(a\) fixed, \((t_{a+h} - t_a)/h\) is a distribution depending on the parameter \(h\). If

\[
\lim_{h \to 0} \frac{t_{a+h} - t_a}{h}
\]

exists, the limiting distribution is denoted by \(dt_a/da\). Thus

\[
\left\langle \frac{dt_a}{da}, \varphi \right\rangle = \lim_{h \to 0} \frac{\left\langle t_{a+h}, \varphi \right\rangle - \left\langle t_a, \varphi \right\rangle}{h}.
\]

The reader should observe that while \(\partial t_a/\partial x_1\), etc., always exist, \(dt_a/da\) exists only if \(\lim_{h \to 0} (t_{a+h} - t_a)/h\) exists.

As an example of differentiation with respect to a parameter, consider \((\partial/\partial \xi_1)\delta(x - \xi)\). With \(e_1\) a unit vector in the \(x_1\) direction,

\[
\left\langle \frac{\partial \delta(x - \xi)}{\partial \xi_1}, \varphi \right\rangle = \lim_{h \to 0} \frac{\left\langle \delta(x - \xi - he_1), \varphi \right\rangle - \left\langle \delta(x - \xi), \varphi \right\rangle}{h}
\]

\[
= \lim_{h \to 0} \frac{\varphi(\xi + he_1) - \varphi(\xi)}{h} = \frac{\partial \varphi}{\partial \xi_1} (\xi).
\]

Since

\[
\left\langle -\frac{\partial \delta(x - \xi)}{\partial x_1}, \varphi \right\rangle = \left(\frac{\partial \varphi}{\partial x_1}\right)_{x = \xi},
\]

we have shown that

\[
\frac{\partial}{\partial \xi_1} \delta(x - \xi) = -\frac{\partial}{\partial x_1} \delta(x - \xi).
\]

Therefore \((\partial/\partial \xi_1)\delta(x - \xi)\) is the volume density arising from a unit dipole at \(\xi\) with axis in the positive \(x_1\) direction.
Example 2. Let \( \sigma \) be a hypersurface in \( R_n \) and let \( b(x) \) be a locally integrable function on \( \sigma \). A normal direction (denoted by \( n \)) is chosen on \( \sigma \) so that \( b \) varies continuously on \( \sigma \). Consider the following functional on \( K_n \):

\[
\langle t, \varphi \rangle = \int_\sigma b(x) \frac{\partial \varphi}{\partial n}(x)dS_x.
\]  

(5.15)

The functional \( t \) is easily seen to be a distribution which vanishes outside \( \sigma \). We can interpret \( t \) in terms of surface sources as follows. Suppose that normally oriented dipoles are spread on \( \sigma \) with surface density \( b(x) \). The surface element \( dS_\xi \) at \( \xi \) carries a dipole whose volume source density is

\[
b(\xi)dS_\xi \frac{\partial}{\partial n_\xi} \delta(x - \xi),
\]

where we have made use of (5.14) to perform the differentiation at the source point. The total volume density is therefore

\[
\int_\sigma b(\xi)dS_\xi \frac{\partial}{\partial n_\xi} \delta(x - \xi).
\]

The action of \( \int_\sigma b(\xi)dS_\xi(\partial/\partial n_\xi)\delta(x - \xi) \) on a test function \( \varphi \) in \( K_n \) is

\[
\left\langle \int_\sigma b(\xi)dS_\xi \frac{\partial}{\partial n_\xi} \delta(x - \xi), \varphi \right\rangle = \int_{R_n} dx \varphi(x) \int_\sigma b(\xi)dS_\xi \frac{\partial}{\partial n_\xi} \delta(x - \xi).
\]

Calculating the integral on the right by first integrating over \( R_n \), we obtain

\[
\left\langle \int_\sigma b(\xi)dS_\xi \frac{\partial}{\partial n_\xi} \delta(x - \xi), \varphi \right\rangle = \int_\sigma b(\xi) \frac{\partial \varphi}{\partial n}(\xi)dS_\xi.
\]

We are therefore entitled to interpret (5.15) as the distribution corresponding to a normally oriented dipole layer spread on \( \sigma \) with surface density \( b(x) \). Such a layer is known as a double layer of strength \( b(x) \).

Example 3. As in Section 1.3, we construct a sequence of functions approaching the delta function (but now in \( n \) dimensions).

Theorem. Let \( f(x) = f(x_1, \ldots, x_n) \) be a nonnegative locally integrable function for which \( \int_{R_n} f(x)dx = 1 \). With \( \alpha > 0 \), define

\[
f_\alpha(x) = \frac{1}{\alpha^n} f\left(\frac{x}{\alpha}\right) = \frac{1}{\alpha^n} f\left(\frac{x_1}{\alpha}, \ldots, \frac{x_n}{\alpha}\right);
\]

then

\[
\lim_{\alpha \to 0} f_\alpha(x) = \delta(x).
\]
Proof. The substitution $u = x/a$ yields the three properties:

(a) $\int_{R_n} f_a(x)dx = 1$;

(b) $\lim_{a \to 0} \int_{|x| > A} f_a(x)dx = 0$ for each $A > 0$;

(c) $\lim_{a \to 0} \int_{|x| < A} f_a(x)dx = 1$ for each $A > 0$.

Our theorem will be proved if it can be shown that, for each test function $\varphi(x)$,

$$\lim_{a \to 0} \int_{R_n} f_a(x)\eta(x)dx = 0,$$

where $\eta(x) = \varphi(x) - \varphi(0)$.

We divide the region of integration $R_n$ into the two parts $|x| < A$ and $|x| > A$, so that

$$\left| \int_{R_n} f_a \eta \, dx \right| \leq \left| \int_{|x| < A} f_a \eta \, dx \right| + \left| \int_{|x| > A} f_a \eta \, dx \right|.$$

Since $\varphi(x)$ is bounded, so is $\eta(x)$; thus $|\eta| \leq M$ for all $x$. Setting $p(A) = \max |\eta|$, and using the nonnegativity of $f$ and property (a), we have

$$\left| \int_{R_n} f_a \eta \, dx \right| \leq p(A) + M \left[ \int_{|x| > A} f_a \, dx \right].$$

We want to show that, for each $\varepsilon > 0$, there exists $\gamma > 0$ such that

$$\left| \int_{R_n} f_a \eta \, dx \right| < \varepsilon,$$

whenever $a < \gamma$.

Since $\eta(x)$ is continuous and $\eta(0) = 0$, $\lim_{A \to 0} p(A) = 0$. We can therefore choose $A$ independently of $a$ such that $p(A) < \varepsilon/2$. With $A$ so chosen, we use property (b) to select $\gamma$ such that

$$\int_{|x| > A} f_a \, dx < \frac{\varepsilon}{2M},$$

whenever $a < \gamma$.

This completes the proof of the theorem.

Example 4. In the special case where $f(x)$ is a function only of $r = (x_1^2 + \cdots + x_n^2)^{1/2}$, that is, $f(x) = g(r)$, the requirement $\int_{R_n} f(x)dx = 1$ of Example 3 reduces to

$$\int_0^\infty r^{s-1}g(r)dr = \frac{1}{S_n(1)},$$
where $S_n(1)$ is the surface area of the unit $n$-dimensional sphere [see (5.19), of Exercise 5.1 below].

Since $S_3(1) = 4\pi$, it follows that for any nonnegative function $g(r)$ satisfying

$$\int_0^\infty r^2 g(r) dr = \frac{1}{4\pi},$$

$$\lim_{\alpha \to 0} \frac{1}{\alpha^3} g\left(\frac{r}{\alpha}\right) = \delta(x_1, x_2, x_3) = \delta(x),$$

where $r = (x_1^2 + x_2^2 + x_3^2)^{1/2}$.

As a particular case we observe that

$$\int_0^\infty r^2 e^{-r^2} dr = \frac{1}{4\pi^{1/2}},$$

so that

$$\lim_{\alpha \to 0} \frac{1}{\alpha^3} (\pi)^{-3/2} \exp\left(-\frac{r^2}{\alpha^2}\right) = \delta(x_1, x_2, x_3) = \delta(x).$$

For further examples, see Exercises 5.2 and 5.3.

**Example 5.** Let $R$ be a region in $R_n$ with boundary $\sigma$ and let

$$u_R = \begin{cases} u, & x \text{ in } R, \\ 0, & x \text{ not in } R, \end{cases}$$

where $u$ is twice differentiable in $R$. Using (5.13), we calculate $\nabla^2 u_R$.

$$\langle \nabla^2 u_R, \varphi \rangle = \langle u_R, \nabla^2 \varphi \rangle = \int_R u \nabla^2 \varphi \, dx = \int_R \varphi \nabla^2 u \, dx + \int_\sigma \left( u \frac{\partial \varphi}{\partial n} - \varphi \frac{\partial u}{\partial n} \right) dS.$$

We note that $\nabla^2 u_R$ consists of the ordinary Laplacian $\nabla^2 u$ plus the sum of a simple layer of surface density $-\partial u(s)/\partial n$ and a double layer of density $u(s)$ on the bounding surface $\sigma$.

**Exercises**

5.1 Let $V_n(r)$ and $S_n(r)$ denote, respectively, the volume and surface area of the $n$-dimensional sphere of radius $r$. To obtain a formula for $V_n(r)$ we first relate it to the volume of an $(n - 1)$-dimensional sphere as follows. Let $z$ denote a coordinate on an axis through the origin. If $A$ is a constant smaller that $r$, the intersection of the hyperplane $z = A$ and the $n$-dimensional sphere is an $(n - 1)$-dimensional sphere of radius $(r^2 - z^2)^{1/2}$. (The reader should convince himself of this by drawing appropriate figures for both the three-dimensional and two-dimensional cases.) Thus

$$V_n(r) = \int_0^r V_{n-1}[(r^2 - z^2)^{1/2}] \, dz.$$
By induction, show that $V_n(r) = C_n r^n$, where

$$C_{n+1} = 2C_n \int_0^1 (1 - u^2)^{n/2} \, du.$$ 

Show that

$$\int_0^1 (1 - u^2)^{n/2} \, du = \frac{\sqrt{\pi} \,(n/2)!}{2 \,[\,(n+1)/2)!]} ,$$

and hence

$$V_n(r) = \frac{\pi^{n/2} r^n}{(n/2)!}.$$  \hspace{1cm} (5.17)

By considering the volume of a thin spherical shell, show that

$$S_n(r) = \frac{nn^{n/2} r^{n-1}}{(n/2)!} = \frac{2\pi^{n/2} r^{n-1}}{[(n/2) - 1]!}.$$ \hspace{1cm} (5.18)

Using $(1/2)! = \sqrt{\pi}/2$, check that the formulas (5.17) and (5.18) are correct for $n = 2$ and $n = 3$.

If $f(x)$ depends only on $r$ [say, $f(x) = g(r)$], show that

$$\int_{K_n} f(x) \, dx = S_n(1) \int_0^\infty g(r) r^{n-1} \, dr.$$ \hspace{1cm} (5.19)

5.2 Show that

$$\lim_{\alpha \to 0} \frac{\alpha}{\pi^2 (r^2 + \alpha^2)^2} = \delta(x_1, x_2, x_3), \quad r = (x_1^2 + x_2^2 + x_3^2)^{1/2};$$

$$\lim_{\alpha \to 0} \frac{\alpha}{2\pi (r^2 + \alpha^2)^{3/2}} = \delta(x_1, x_2), \quad r = (x_1^2 + x_2^2)^{1/2}.$$

5.3 From elementary calculus, we have, for $a > 0$,

$$\int_0^\infty e^{-ar^2} \, dr = \frac{1}{2} \left( \frac{\pi}{a} \right)^{1/2}, \quad \int_0^\infty re^{-ar^2} \, dr = \frac{1}{2a}.$$

Differentiating these expressions repeatedly with respect to $a$, show that

$$\int_0^\infty r^{2n} e^{-ar^2} \, dr = \left( \frac{\pi}{a} \right)^{1/2} a^{-n-2} 2^n (2n-1)! \, (n-1)!,$$

$$\int_0^\infty r^{2n+1} e^{-ar^2} \, dr = \frac{n!}{2a^{n+1}}.$$

Consult Exercise 5.1 to show that, for any positive integer $n$,

$$\lim_{\alpha \to 0} \frac{1}{\alpha^n} e^{-r^2/\alpha^2} = (\pi)^{n/2} \delta(x_1, \ldots, x_n),$$ \hspace{1cm} (5.20)

where $r = (x_1^2 + \cdots + x_n^2)^{1/2}$. 

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5.4 Let \( f_\alpha(x) = f_\alpha(x_1, \ldots, x_n) \) be a set of nonnegative locally integrable functions of \( x \) with the properties:

(a) For all \( A \) and \( B \) such that \( 0 < A < B < \infty \),

\[
\lim_{\alpha \to 0} \int_{A < |x| < B} f_\alpha(x) \, dx = 0;
\]

(b) For each \( A > 0 \),

\[
\lim_{\alpha \to 0} \int_{|x| < A} f_\alpha(x) \, dx = 1.
\]

Prove that

\[
\lim_{\alpha \to 0} f_\alpha(x) \, dx = \delta(x).
\]

5.5 Consider the sequence of locally integrable functions of the single independent variable \( x \),

\[
f_\alpha(x) = \begin{cases} 
0, & x < 0; \\
\alpha e^{-\frac{x^2}{4\alpha}} \sqrt{4\pi x^{3/2}}, & x > 0.
\end{cases}
\]

Using the result of Exercise 5.4, show that

\[
\lim_{\alpha \to 0} f_\alpha(x) = \delta(x).
\]

5.6 Let \( f(x) \) be an even nonnegative function of the real variable \( x \) for which \( \int_{-\infty}^{\infty} f(x) \, dx = 1 \). Show that if \( u(x) \) is an arbitrary, bounded, locally integrable function for which \( u(0+) \) and \( u(0-) \) exist, then

\[
\lim_{\alpha \to 0^+} \int_{-\infty}^{\infty} \frac{1}{\alpha} f\left( \frac{x}{\alpha} \right) u(x) \, dx = \frac{u(0^+) + u(0^-)}{2}.
\]

5.7 Let \( f(x) \) be a nonnegative function for which \( \int_{0}^{\infty} f(x) \, dx = A \). Show that if \( u(x) \) is an arbitrary, bounded, locally integrable function for which \( u(0^+) \) exists, then

\[
\lim_{\alpha \to 0^+} \int_{0}^{\infty} \frac{1}{\alpha} f\left( \frac{x}{\alpha} \right) u(x) \, dx = Au(0^+).
\]

5.8 If \( x \) is a real variable, \( -\infty < x < \infty \), show that

\[
\lim_{R \to \infty} \frac{\sin^2 Rx}{\pi Rx^2} = \delta(x).
\]
5.5 ADDITIONAL PROPERTIES OF DISTRIBUTIONS

Direct Product of Distributions

Let $t_1$ and $t_2$ be one-dimensional distributions. The direct or Cartesian product $t_1(x_1)t_2(x_2)$ is a two-dimensional distribution which is formally defined as

$$\langle t_1(x_1)t_2(x_2), \varphi(x_1, x_2) \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} t_1(x_1)t_2(x_2), \varphi(x_1, x_2) \, dx_1 \, dx_2. \quad (5.21)$$

The symbolic operations on the right side must be defined precisely in terms of the actions of $t_1$ and $t_2$. If $x_1$ is kept constant, $\varphi(x_1, x_2)$ is a one-dimensional test function in $x_2$ so that $\langle t_2(x_2), \varphi(x_1, x_2) \rangle$ is well defined and depends parametrically on $x_1$. Setting

$$\psi(x_1) = \langle t_2(x_2), \varphi(x_1, x_2) \rangle,$$

we easily see that $\psi(x_1)$ is a test function in $x_1$. Hence we define

$$\langle t_1(x_1)t_2(x_2), \varphi(x_1, x_2) \rangle = \langle t_1(x_1), \psi(x_1) \rangle$$

$$= \langle t_1(x_1), \langle t_2(x_2), \varphi(x_1, x_2) \rangle \rangle.$$

We can verify without difficulty that $t_2(x_2)t_1(x_1) = t_1(x_1)t_2(x_2)$.

EXAMPLES

Example 1. In $R_2$,

$$\delta(x) = \delta(x_1)\delta(x_2).$$

Here $\delta(x)$ is the two-dimensional delta function and $\delta(x_1)\delta(x_2)$ is the direct product of two one-dimensional delta functions.

To prove the above assertion we observe that, by definition,

$$\langle \delta(x), \varphi(x_1, x_2) \rangle = \varphi(0, 0),$$

whereas

$$\langle \delta(x_1)\delta(x_2), \varphi(x_1, x_2) \rangle = \langle \delta(x_1), \langle \delta(x_2), \varphi(x_1, x_2) \rangle \rangle$$

$$= \langle \delta(x_1), \varphi(x_1, 0) \rangle = \varphi(0, 0).$$

Example 2. In $R_n$,

$$\delta(x) = \delta(x_1)\delta(x_2) \cdots \delta(x_n).$$
Example 3. In $R_3$, consider the direct product of $\delta(x_1)$ and a locally integrable function of the variables $x_2$ and $x_3$:

$$\langle \delta(x_1), f(x_2, x_3), \varphi(x_1, x_2, x_3) \rangle$$

$$= \langle \delta(x_1), \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(x_1, x_2, x_3)f(x_2, x_3)dx_2 \, dx_3 \rangle$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(0, x_2, x_3)f(x_2, x_3)dx_2 \, dx_3.$$  

Thus $\delta(x_1)f(x_2, x_3)$ is the volume source density corresponding to a simple layer of sources of surface density $f(x_2, x_3)$ spread on the plane $x_1 = 0$. In particular, $\delta(x_1)1(x_2, x_3)$, where $1(x_2, x_3)$ is the function identically 1, corresponds to a simple layer of unit surface density on the plane $x_1 = 0$. By an abuse of notation we sometimes write $\delta(x_1)1(x_2, x_3)$ as $\delta(x_1)$. The context should make it clear whether $\delta(x_1)$ is being interpreted as one-dimensional distribution or as the three-dimensional distribution $\delta(x_1)1(x_2, x_3)$. From the first point of view, $\delta(x_1)$ acts on one-dimensional test functions and

$$\langle \delta(x_1), \varphi(x_1) \rangle = \varphi(0),$$

whereas from the second point of view $\delta(x_1)$ acts on three-dimensional test functions:

$$\langle \delta(x_1), \varphi(x_1, x_2, x_3) \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(0, x_2, x_3)dx_2 \, dx_3.$$  

**Convolution Product**

If $f_1(x)$ and $f_2(x)$ are locally integrable functions on the line, for which

$$\int_{-\infty}^{\infty} |f_1(x)|dx \quad \text{and} \quad \int_{-\infty}^{\infty} |f_2(x)|dx$$

are both finite, we define their *convolution* $h(x)$ by

$$h(x) = f_1 * f_2 = \int_{-\infty}^{\infty} f_1(x - y)f_2(y)dy = \int_{-\infty}^{\infty} f_2(x - y)f_1(y)dy = f_2 * f_1.$$  

(5.22)

It is easily ascertained that $h(x)$ is locally integrable and therefore defines a distribution which we would like to express in terms of the distributions $f_1$ and $f_2$. We have

$$\langle h, \varphi \rangle = \int_{-\infty}^{\infty} \varphi(x)dx \int_{-\infty}^{\infty} f_1(x - y)f_2(y)dy = \int_{-\infty}^{\infty} f_2(y)\psi(y)dy,$$

$$\langle h, \varphi \rangle = \int_{-\infty}^{\infty} \varphi(x)dx \int_{-\infty}^{\infty} f_2(x - y)f_1(y)dy = \int_{-\infty}^{\infty} f_1(y)\chi(y)dy,$$
where

\[ \psi(y) = \int_{-\infty}^{\infty} f_1(x - y)\varphi(x)dx = \int_{-\infty}^{\infty} f_1(z)\varphi(y + z)dz, \]

\[ \chi(y) = \int_{-\infty}^{\infty} f_2(z)\varphi(y + z)dz. \]

Although \( \psi \) and \( \chi \) are infinitely differentiable, they are not, in general, test functions. However if either \( f_1 \) or \( f_2 \) vanishes outside a finite interval, then \( \psi \) or \( \chi \) will itself vanish outside a finite interval. Suppose, for instance, that \( f_1(x) = 0 \) for \( |x| > a \) and \( \varphi(x) = 0 \) for \( |x| > b \); then \( \psi(y) = 0 \) for \( |y| > a + b \). Thus if \( f_1 \) vanishes outside a finite interval, we define \( h \) from

\[ \langle h, \varphi \rangle = \langle f_2(x), \langle f_1(z), \varphi(x + z) \rangle \rangle, \]

and if \( f_2 \) vanishes outside a finite interval,

\[ \langle h, \varphi \rangle = \langle f_1(x), \langle f_2(z), \varphi(x + z) \rangle \rangle. \]

These definitions clearly make sense even when \( f_1 \) and \( f_2 \) are distributions at least one of which vanishes outside a finite interval. Moreover, there is no essential restriction as to the number of dimensions. Thus if \( t_1 \) and \( t_2 \) are both distributions in \( n \) variables and \( t_1 \) vanishes outside a finite interval, we define the convolution product \( h = t_1 \ast t_2 \) as the \( n \)-dimensional distribution

\[ \langle h, \varphi \rangle = \langle t_2(x), \langle t_1(z), \varphi(x + z) \rangle \rangle, \quad (5.23) \]

**Examples**

**Example 1.** Consider the convolution of \( \delta(x) \) and \( t(x) \), where \( \delta \) is the \( n \)-dimensional Dirac distribution and \( t \) is an arbitrary \( n \)-dimensional distribution. Then from (5.23),

\[ \langle \delta \ast t, \varphi \rangle = \langle t(x), \langle \delta(z), \varphi(x + z) \rangle \rangle = \langle t(x), \varphi(x) \rangle. \]

Thus

\[ \delta \ast t = t. \quad (5.24) \]

**Example 2.** Let \( L \) be a differential operator of order \( p \). We want to calculate \((L\delta) \ast t\), where \( t \) is an arbitrary distribution. We have†

\[ \langle (L\delta) \ast t, \varphi \rangle = \langle t(x), \langle L\delta(z), \varphi(x + z) \rangle \rangle = \langle t(x), \langle \delta(z), L^* \varphi(x + z) \rangle \rangle \]

\[ = \langle t(x), L^* \varphi(x) \rangle = \langle Lt, \varphi \rangle. \]

Thus

\[ (L\delta) \ast t = Lt. \quad (5.25) \]

† When the symbol \( ^* \) is used as a superscript, it denotes the adjoint as in (5.12).
Example 3. Let $L$ be a differential operator with constant coefficients, and let $s$ and $t$ be two distributions on $K_n$, with $s$ vanishing outside a finite sphere. We want to show

$$(Ls) * t = s * (Lt) = L(s * t).$$

We have

$$
\langle L(s * t), \varphi \rangle = \langle s * t, L^* \varphi \rangle \\
= \langle t(x), \langle s(z), L^* \varphi(x + z) \rangle \rangle \\
= \langle t(x), \langle Ls(z), \varphi(x + z) \rangle \rangle \\
= \langle (Ls) * t, \varphi \rangle.
$$

The rest of the proof is left to the reader.

**Transformation Properties of the Delta Function**

One is sometimes required to interpret the expression $\delta[f(x)]$. The equivalent functional is $\int_{-\infty}^{\infty} \delta[f(x)] \varphi(x) dx$. Suppose first that $f(x)$ has a simple zero at $x = x_0$ with $f''(x_0) > 0$; the function $u = f(x)$ is therefore one-to-one in a neighborhood of $x = x_0$. Making the substitution $u = f(x)$ in the integral, we obtain $\int \delta(u) \varphi[x(u)](dx/du)du$. Only a neighborhood of $u = 0$ contributes to the integral, and the integration is in the direction of increasing $u$. The value of the integral is therefore $[1/f''(x_0)] \varphi(x_0)$. If, on the other hand, we have $f''(x_0) < 0$, the direction of integration is reversed and we obtain $-[1/f''(x_0)] \varphi(x_0)$. Both results are incorporated in the formula

$$
\delta[f(x)] = \frac{\delta(x - x_0)}{|f''(x_0)|}.
$$

If the function $f(x)$ has simple zeros at $x_1, \ldots, x_k$, then

$$
\delta[f(x)] = \sum_{n=1}^{k} \frac{\delta(x - x_n)}{|f''(x_n)|}.
$$

**Examples**

Example 1. $\delta(x^2 - a^2) = \frac{1}{2a} [\delta(x - a) + \delta(x + a)].$

Example 2. $\delta(\sin x) = \sum_{n=-\infty}^{\infty} \delta(x - n\pi).$

Example 3. If $f(x)$ has a double or higher zero, we make no attempt to attach a significance to $\delta[f(x)]$. 

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Next we would like to express the \( n \)-dimensional Dirac distribution \( \delta(x - x') \) in a new coordinate system. Let the new coordinates \( u_1, \ldots, u_n \) be obtained from \( x_1, \ldots, x_n \) by the transformation law \( u = u(x) \); that is, \( u_1 = u_1(x_1, \ldots, x_n), \ldots, u_n = u_n(x_1, \ldots, x_n) \). We shall assume that the Jacobian of the transformation does not vanish at \( x' \), and that the inverse transformation \( x(u) \) exists. The test function \( \varphi(x) \) may be expressed either as \( \varphi(x_1, \ldots, x_n) \) or \( \tilde{\varphi}(u_1, \ldots, u_n) \), where

\[
\tilde{\varphi}(u_1, \ldots, u_n) = \varphi[x_1(u_1, \ldots, u_n), \ldots, x_n(u_1, \ldots, u_n)].
\]

From the definition of \( \delta(x - x') \), we have

\[
\varphi(x_1', \ldots, x_n') = \int_{R^n} \delta(x - x')\varphi(x_1, \ldots, x_n)dx_1 \cdots dx_n
\]

or

\[
\varphi(x_1', \ldots, x_n') = \int_{u\text{ space}} \delta[x(u) - x']\tilde{\varphi}(u_1, \ldots, u_n)J\, du
\]

where \( J \) is the Jacobian of the \( x \) variables with respect to the \( u \) variables. Setting \( u_i' = u_i(x_1', \ldots, x_n') \), \( i = 1, \ldots, n \), we have

\[
\tilde{\varphi}(u_1', \ldots, u_n') = \int_{u\text{ space}} \delta[x(u) - x']\tilde{\varphi}(u_1, \ldots, u_n)J\, du,
\]

from which we infer

\[
\delta[x(u) - x']|J| = \delta(u_1 - u_1') \cdots \delta(u_n - u_n'),
\]

where the absolute value sign is needed to compensate for possible reversals of the direction of integration in \( u \) space. Thus

\[
\delta(x_1 - x_1') \cdots \delta(x_n - x_n') = \frac{1}{|J|} \delta(u_1 - u_1') \cdots \delta(u_n - u_n'). \tag{5.29}
\]

This relation is valid only if \( J \neq 0 \) at \( x' \). In the example which follows we show the modifications needed when \( J \) vanishes at \( x' \).

**EXAMPLE**

Let \( u_1, u_2, \) and \( u_3 \) be spherical coordinates in \( R_3 \) (usually denoted by \( r, \theta, \) and \( \varphi \), respectively). Then \( J = u_1^2 \sin u_2 \), and

\[
\delta(x - x') = \frac{1}{u_1^2 \sin u_2} \delta(u_1 - u_1')\delta(u_2 - u_2')\delta(u_3 - u_3'). \tag{5.30}
\]

The points lying on the \( x_3 \) axis are singular points of the transformation and we must alter the procedure. Consider first \( \delta(x_1, x_2, x_3) \). Because of the spherical symmetry, one expects that, when expressed in spherical
coordinates, \( \delta(x_1, x_2, x_3) \) will give rise to a symbolic function \( \eta \) which depends only on \( u_1 \). In fact, we have

\[
\varphi(0, 0, 0) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x_1, x_2, x_3) \varphi(x_1, x_2, x_3) \, dx_1 \, dx_2 \, dx_3 \\
= \int_0^{\infty} du_1 \int_0^{\pi} du_2 \int_0^{2\pi} du_3 \eta(u_1) u_1^2 \sin u_2 \, \bar{\varphi}(u_1, u_2, u_3).
\]

We note that \((1/4\pi) \int_0^{\pi} du_2 \int_0^{2\pi} du_3 \sin u_2 \, \bar{\varphi}(u_1, u_2, u_3)\) is the average of \( \varphi \) on a sphere of radius \( u_1 \) with center at the origin. Therefore, \( \varphi(0, 0, 0) = 4\pi \int_0^{\infty} du_1 \eta(u_1) u_1^2 \varphi_{av}(u_1) \); as \( u_1 \to 0 \), \( \varphi_{av}(u_1) \to \varphi(0, 0, 0) \), so that

\[
\eta(u_1) = \frac{\delta(u_1)}{4\pi u_1^2}.
\]

Hence

\[
\delta(x) = \frac{\delta(u_1)}{4\pi u_1^2},
\]

where \( u_1 = (x_1^2 + x_2^2 + x_3^2)^{1/2} \). It is important to realize that in performing an integration from 0 to \( \infty \) on \( \delta(u_1) \), the entire source term is included.

In a similar way, one can show that if \( u_2' = 0 \) (positive \( x_3 \) axis),

\[
\delta(x - x') = \frac{\delta(u_1 - u_1') \delta(u_2)}{2\pi u_1^2 \sin u_2}
\]

(5.31)

**Exercises**

5.9 Derive the form for \( \delta(x - x') \) in spherical coordinates, when \( x_3' > 0 \), \( x_2' = x_1' = 0 \); find the appropriate expression for \( x_3' < 0 \), \( x_2' = x_1' = 0 \).

5.10 Find the form for \( \delta(x) \) in spherical coordinates in \( \mathbb{R}_n \). The source is located at the origin of the coordinate system.

5.11 Consider sources spread with uniform surface density \( 1/4\pi \varepsilon^2 \) on the surface of the sphere \( x^2 + y^2 + z^2 = \varepsilon^2 \) in \( \mathbb{R}_3 \). The resulting volume source density is

\[
\int_{\sigma} \frac{1}{4\pi \varepsilon^2} \delta(x - \xi) \, dS_\xi,
\]

where \( \xi \) is a point on the surface \( \sigma \) of the sphere. Introduce spherical coordinates to show that

\[
\int_{\sigma} \frac{1}{4\pi \varepsilon^2} \delta(x - \xi) \, dS_\xi = \frac{\delta(r - \varepsilon)}{4\pi \varepsilon^2} = \frac{\delta(u_1 - \varepsilon)}{4\pi u_1^2}.
\]

In the limit as \( \varepsilon \to 0 \) we have a unit source at the origin.
Again we consider spherical coordinates in $R^3$. Let a ring of sources be located on the intersection of the sphere $u_1 = r_0$ and the cone $u_2 = \theta_0$. This intersection is a circle of radius $r_0 \sin \theta_0$. We take the line density of the source ring to be $1/2\pi r_0 \sin \theta_0$, so that the total strength of the source is unity. (Observe that, for $\theta_0$ near 0, this approximates a point source on the $x_3$ axis.) Show that the volume density is given by

$$\frac{\delta(u_1 - r_0)\delta(u_2 - \theta_0)}{2\pi u_1^2 \sin u_2}.$$ 

Therefore as $\theta_0 \to 0$, we obtain (5.31).

Show that it is possible to define the convolution of two distributions on the real line if they both vanish for $x < 0$.

Show that if $ad - bc \neq 0$,

$$\delta(ax_1 + bx_2)\delta(cx_1 + dx_2) = \frac{1}{|ad - bc|} \delta(x_1)\delta(x_2).$$

Classical Results

Let $f(x)$ be a complex-valued function on the real line, $-\infty < x < \infty$. The Fourier transform of $f$ is a function $\hat{f}$ defined from

$$f^\wedge(u) = \int_{-\infty}^{\infty} e^{iux}f(x)dx,$$  \hspace{1cm} (5.32)

where at first the so-called transform variable $u$ is taken real. The integral (5.32) will not exist for every $f(x)$; in fact, for as simple a function as $f(x) = 1$, the integral fails to converge for every $u$. However, it is easily established that $f^\wedge(u)$ will exist for every real value of $u$ if $f(x)$ is in $\mathcal{L}(-\infty, \infty)$, that is, if

$$\int_{-\infty}^{\infty} |f(x)|dx < \infty.$$  \hspace{1cm} (5.33)

If, in addition to (5.33), $f(x)$ satisfies certain mild conditions—for instance, if $f(x)$ is piecewise smooth—we can recover $f$ from $f^\wedge$ by an inversion formula known as the Fourier integral theorem:

$$f(x) = \lim_{R \to \infty} \frac{1}{2\pi} \int_{-R}^{R} f^\wedge(u)e^{-iux} du.$$  \hspace{1cm} (5.34)

We shall prove the theorem under the assumption that $f^\prime(x)$ is continuous. We must show that

$$\lim_{R \to \infty} f_R(x) = f(x),$$

We shall prove the theorem under the assumption that $f^\prime(x)$ is continuous. We must show that
where

\[ f_R(x) = \frac{1}{2\pi} \int_{-R}^{R} f^\wedge(u)e^{-iux} \, du. \]

From this definition of \( f_R(x) \), we have

\[
\begin{align*}
f_R(x) &= \frac{1}{2\pi} \int_{-R}^{R} e^{-iux} \, du \int_{-\infty}^{\infty} e^{iuy} f(y) \, dy \\
&= \int_{-\infty}^{\infty} \frac{\sin R(y-x)}{\pi(y-x)} f(y) \, dy = \int_{-\infty}^{\infty} \frac{\sin Rz}{\pi z} f(x+z) \, dz.
\end{align*}
\]

Since \( f' \) is continuous, the result of Exercise 1.18(b) shows that the last integral tends to \( f(x) \) as \( R \to \infty \); stated in different terms, the sequence \( \sin Rz/\pi z \) is a delta sequence.

The inversion formula (5.34) can be proved under less restrictive conditions on \( f \). It suffices for \( f(x) \) to be piecewise continuous and have a piecewise continuous derivative as long as we interpret the left side of (5.34) as \([f(x+)+f(x-)]/2\).

For further reference we mention a number of relations involving Fourier transforms. All these relations go under the name Parseval's formulas:

\[
\begin{align*}
\int_{-\infty}^{\infty} f^\wedge(x)g(x) \, dx &= \int_{-\infty}^{\infty} f(x)g^\wedge(x) \, dx, \\
\int_{-\infty}^{\infty} f(x)g(x) \, dx &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f^\wedge(u)g^\wedge(-u) \, du, \\
\int_{-\infty}^{\infty} f(x)\overline{g}(x) \, dx &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f^\wedge(u)\overline{g^\wedge(u)} \, du, \\
\int_{-\infty}^{\infty} |f(x)|^2 \, dx &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |f^\wedge(u)|^2 \, du.
\end{align*}
\]

The Parseval formulas are easily derived from the definition of the Fourier transform by formally interchanging orders of integration. We shall not at present state precise conditions for the validity of these formulas.

Equations (5.32) and (5.34) can be generalized to some extent by allowing the transform variable to take on complex values. Renaming the transform variable \( \omega = u + iv \), where \( u \) and \( v \) are real,

\[
\begin{align*}
f^\wedge(\omega) &= f^\wedge(u + iv) = \int_{-\infty}^{\infty} e^{iux} f(x) \, dx \\
&= \int_{-\infty}^{\infty} e^{iux} e^{-vx} f(x) \, dx.
\end{align*}
\]


Thus $f^\wedge(\omega)$ is just the Fourier transform (5.32) of the function $e^{-vxf(x)}$. If $v$ is chosen so that $e^{-vxf}(x)$ is in $\mathcal{L}(-\infty, \infty)$, we obtain from (5.34),

$$e^{-vxf}(x) = \lim_{R \to \infty} \frac{1}{2\pi} \int_{-R}^{R} f^\wedge(u + iv)e^{-iux} \, du.$$  

This last integral can be interpreted as an integral in the complex $\omega$ plane along a line parallel to the real axis. In fact, we have

$$f(x) = \lim_{R \to \infty} \frac{1}{2\pi} \int_{iv-R}^{iv+R} f^\wedge(\omega)e^{-iux} \, d\omega,$$

or, by a slight abuse of notation,

$$f(x) = \frac{1}{2\pi} \int_{iv-\infty}^{iv+\infty} f^\wedge(\omega)e^{-iux} \, d\omega. \tag{5.40}$$

In (5.40) the value of $v$ is any real number for which

$$\int_{-\infty}^{\infty} |e^{-vxf}(x)| \, dx < \infty. \tag{5.41}$$

One should observe that the factor $e^{-vxf}$ which occurs in (5.41) is not necessarily helpful; if $v > 0$, the factor improves convergence at the upper limit but impairs it at the lower limit, and conversely for $v < 0$. Even in the simple case $f(x) = 1$, there is no value of $v$ for which (5.41) holds, so that our approach will have to be modified. On the other hand, there are cases when the inequality (5.41) is satisfied in an entire strip $v_1 < v < v_2$; it can then be shown that $f^\wedge(\omega)$ is an analytic function in that strip.

**EXAMPLES**

Example 1. $f(x) = e^{-|x|}$. Then $f(x)e^{-vxf}$ is in $\mathcal{L}(-\infty, \infty)$, for all $v$ such that $-1 < v < 1$. We find

$$f^\wedge(\omega) = \int_{-\infty}^{\infty} e^{i\omega x}e^{-x} \, dx + \int_{0}^{\infty} e^{i\omega x}e^{-x} \, dx$$

$$= \frac{1}{1 + i\omega} + \frac{1}{1 - i\omega} = \frac{2}{1 + \omega^2}, \quad -1 < v < 1.$$  

We now illustrate how (5.34), or (5.40) with $v = 0$, can be used to recover $f(x)$ from $f^\wedge(\omega)$ by the method of contour integration. For $x > 0$, $2e^{-i\omega x}/(1 + \omega^2)$ is exponentially small in the lower half of the $\omega$ plane ($\omega < 0$). Consider the contour $C_R$ consisting of the entire boundary of the large semicircle shown in Figure 5.1. Then by Cauchy's theorem,

$$\lim_{R \to \infty} \frac{1}{2\pi} \int_{C_R} \frac{2}{1 + \omega^2} e^{-i\omega x} \, d\omega = 2\pi i r,$$
where $r$ is the sum of the residues of

$$\frac{1}{\pi(1 + \omega^2)} e^{-i\omega x}$$

within $C_R$. The only singularity is a simple pole at $\omega = -i$ and the corresponding residue is

$$-\frac{1}{2i\pi} e^{-x}.$$

Now, as $R \to \infty$, the contribution from the curved portion of $C_R$ disappears because of the behavior of the integrand, whereas the contribution from the diameter is

$$-\int_{-\infty}^{\infty} \frac{1}{2\pi} \frac{2}{1 + \omega^2} e^{-i\omega x} d\omega.$$

We therefore find that

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2}{1 + \omega^2} e^{-i\omega x} d\omega = e^{-x}, \quad x > 0.$$

Similarly, by considering a semicircle in the upper half-plane, one easily shows that the value of the inversion integral is $e^x$ for $x < 0$.

**Example 2.** $f(x) = -2H(x) \sinh x$, where $H(x)$ is the Heaviside function. Thus

$$f(x) = \begin{cases} -2 \sinh x, & x > 0; \\ 0, & x < 0. \end{cases}$$
Since $2 \sinh x = e^x - e^{-x}$, we see that $e^{-ux} \sinh x$ is in $\mathscr{L}(0, \infty)$ for $v > 1$, and therefore $e^{-ux}f(x)$ is in $\mathscr{L}(-\infty, \infty)$ for $v > 1$. A simple calculation yields

$$f^\wedge(\omega) = \int_0^\infty (e^{-x} - e^x)e^{i\omega x} \, dx = \frac{2}{1 + \omega^2}, \quad v > 1.$$  

When compared with the result of Example 1, this may at first seem mystifying. Do two different functions have the same Fourier transform? The answer is that transforms of different functions may have the same functional form valid in different, nonoverlapping regions of the $\omega$ plane. We can still use the inversion formula (5.40), this time with $v > 1$, to recover the original $f(x)$ of our present example. First, for $x < 0$, we use a semicircle in the region $v > 1$ to find $f = 0$ for $x < 0$. For $x > 0$, we use a lower semicircle, which now includes both poles of the integrand (at $\omega = +i$ and $\omega = -i$) and obtain $f = -2 \sinh x$.

**Example 3.** $f(x) = e^{-x^2}$. Then $fe^{-ux}$ is in $\mathscr{L}(-\infty, \infty)$ for all real $v$, so that $f^\wedge(\omega)$ is analytic in the whole $\omega$ plane. We find

$$f^\wedge(\omega) = \int_{-\infty}^\infty e^{i\omega x} e^{-x^2} \, dx = e^{-\omega^2/4} \int_{-\infty}^\infty e^{-(x-i\omega/2)^2} \, dx = \sqrt{\pi} e^{-\omega^2/4}.$$  

**Example 4.** Let $\theta$ be a given positive constant; consider

$$f_\theta(x) = \begin{cases} 1/2\theta, & |x| < \theta; \\ 0, & |x| > \theta. \end{cases}$$

Clearly $f_\theta e^{-ux}$ is in $\mathscr{L}(-\infty, \infty)$ for all real $v$, so that $f_\theta^\wedge(\omega)$ will be analytic in the entire $\omega$ plane. We have

$$f_\theta^\wedge(\omega) = \frac{1}{2\theta} \int_{-\theta}^\theta e^{i\omega x} \, dx = \frac{\sin \omega \theta}{\omega \theta},$$

which is, in fact, analytic in the entire $\omega$ plane including $\omega = 0$. The inversion formula can be used on the real axis of the $\omega$ plane, so that

$$f_\theta(x) = \frac{1}{2\pi} \int_{-\infty}^\infty \frac{\sin u\theta}{u\theta} e^{-iux} \, du.$$  

Without attention to rigor, let us examine the limit as $\theta \to 0$. Then $f_\theta(x) \to \delta(x)$ and $\sin u\theta/u\theta \to 1$. Thus we surmise that in some appropriate sense,

$$\delta^\wedge(u) = 1,$$

$$1^\wedge(u) = 2\pi \delta(u).$$

These formulas will be shown to hold rigorously in the distributional sense.
One-Sided Functions

A function which vanishes for \( x < 0 \) is said to be right-sided (or causal) and will be denoted by \( f_+(x) \); a function which vanishes for \( x > 0 \) is said to be left-sided and will be denoted by \( f_-(x) \).

Consider a right-sided function \( f_+ \) which is \( 0(e^{\beta x}) \) at \( x = +\infty \), that is, such that there exists a constant \( C \) with the property

\[
\left| \frac{f_+(x)}{e^{\alpha x}} \right| < C \quad \text{for } x \text{ sufficiently large.}
\]

Then \( f_+(x)e^{-\alpha x} \) is in \( \mathcal{L}(-\infty, \infty) \) for \( \nu > \alpha \) and therefore the Fourier transform \( f_+^\wedge(\omega) \) of \( f_+(x) \) is analytic in the upper half-plane \( \nu > \alpha \). The formulas (5.39) and (5.40) then become

\[
f_+^\wedge(\omega) = \int_0^\infty f_+(x)e^{i\omega x} \, dx, \quad \nu > \alpha; \tag{5.42}
\]

\[
f_+(x) = \frac{1}{2\pi} \int_{i\nu-\infty}^{i\nu+\infty} f_+^\wedge(\omega)e^{-i\omega x} \, d\omega, \quad \nu > \alpha, \tag{5.43}
\]

respectively. In particular, this implies that the integral in (5.43) vanishes identically for \( x < 0 \).

For a left-sided function \( f_- \) which is \( 0(e^{\beta x}) \) at \( x = -\infty \), \( f_-(x)e^{-\alpha x} \) belongs to \( \mathcal{L}(-\infty, \infty) \) for \( \nu < \beta \) and the Fourier transform \( f_-^\wedge(\omega) \) of \( f_-(x) \) is analytic in the lower half-plane \( \nu < \beta \). We therefore have the transform relations

\[
f_-^\wedge(\omega) = \int_{-\infty}^{0} f_-(x)e^{i\omega x} \, dx, \quad \nu < \beta; \tag{5.44}
\]

\[
f_-(x) = \frac{1}{2\pi} \int_{i\nu-\infty}^{i\nu+\infty} f_-^\wedge(\omega)e^{-i\omega x} \, d\omega, \quad \nu < \beta. \tag{5.45}
\]

The integral in (5.45) vanishes identically for \( x > 0 \).

Now, if \( f(x) \) is an arbitrary function on the real line, we can write

\[
f(x) = f_+(x) + f_-(x),
\]

where

\[
f_+(x) = \begin{cases} f(x), & x > 0, \\ 0, & x < 0, \end{cases} \quad f_-(x) = \begin{cases} 0, & x > 0, \\ f(x), & x < 0. \end{cases}
\]

If \( f(x) \) is \( 0(e^{\beta x}) \) at \( x = +\infty \) and \( 0(e^{\alpha x}) \) at \( x = -\infty \), we have, by combining our previous results,

\[
f_+^\wedge(\omega) = \int_0^\infty f(x)e^{i\omega x} \, dx, \quad \nu > \alpha, \tag{5.46}
\]

\[
f_-^\wedge(\omega) = \int_{-\infty}^{0} f(x)e^{i\omega x} \, dx, \quad \nu < \beta, \tag{5.47}
\]

\[
f(x) = \frac{1}{2\pi} \int_{i\nu-\infty}^{i\nu+\infty} f_+^\wedge(\omega)e^{-i\omega x} \, d\omega + \frac{1}{2\pi} \int_{i\beta-\infty}^{i\beta+\infty} f_-^\wedge(\omega)e^{-i\omega x} \, d\omega, \tag{5.48}
\]
where \( a > \alpha \) and \( b < \beta \). These formulas provide a useful generalization of (5.39) and (5.40). If it happens that \( \beta > \alpha \), the Fourier transform \( f^\wedge(\omega) \) exists for \( \alpha < \nu < \beta \) and we can choose \( a = b \) in this strip to reduce (5.48) to (5.40).

**Functions of Slow Growth**

**Definition.** A function \( f(x) \) on the real line is said to be of slow growth if

1. \( f \) is locally integrable, that is, \( \int_I |f(x)| dx \) is finite for each bounded interval \( I \).
2. There exist constants \( C, n, \) and \( R \) such that
   \[
   |f(x)| < C|x|^n \quad \text{for} \quad |x| > R.
   \]

Thus a function of slow growth is one which grows at infinity more slowly than some polynomial. Of course a function \( f(x) \) of slow growth does not usually have a Fourier transform in the sense of (5.32) or (5.39), but \( f^+(x)e^{-\nu x} \) is in \( \mathcal{L}(-\infty, \infty) \) for each \( \nu > 0 \) and \( f^-(x)e^{\nu x} \) is in \( \mathcal{L}(-\infty, \infty) \) for each \( \nu < 0 \). Hence we can use (5.48) for any \( a > 0 \) and \( b < 0 \), and therefore

\[
f(x) = \frac{1}{2\pi} \lim_{\varepsilon \to 0^+} \left[ \int_{-\infty}^{\varepsilon+i\infty} f^+(\omega)e^{-i\omega x} d\omega + \int_{-\varepsilon-i\infty}^{-i\infty} f^-(\omega)e^{-i\omega x} d\omega \right]
\]

or

\[
f(x) = \frac{1}{2\pi} \lim_{\varepsilon \to 0^+} \int_{-\infty}^{\infty} e^{-ix\nu}[e^{\varepsilon\nu}f^+(u+i\varepsilon) + e^{-\varepsilon\nu}f^-(u-i\varepsilon)] du.
\]

One would hope that in some appropriate sense we could say that \( f^\wedge(u) \) exists and that

\[
f^\wedge(u) = \lim_{\varepsilon \to 0^+} [f^\wedge(u+i\varepsilon) + f^\wedge(u-i\varepsilon)] = \lim_{\varepsilon \to 0^+} \int_{-\infty}^{\infty} f(x)e^{i\varepsilon x}e^{-\varepsilon|x|} dx.
\]

Such an interpretation will be shown to be possible in the theory of distributions (see Exercise 5.17). At present we content ourselves with a simple example. Let \( f(x) = 1, -\infty < x < \infty \); then \( f(x) \) is clearly of slow growth and

\[
f^\wedge(u+i\varepsilon) = \int_{0}^{\infty} e^{i(u+i\varepsilon)x} dx = \frac{i}{u+i\varepsilon},
\]

\[
f^\wedge(u-i\varepsilon) = \int_{-\infty}^{0} e^{i(u-i\varepsilon)x} dx = -\frac{i}{u-i\varepsilon},
\]

Thus

\[
f^\wedge(u+i\varepsilon) + f^\wedge(u-i\varepsilon) = \frac{2\varepsilon}{u^2 + \varepsilon^2},
\]
and, as was shown in Chapter 1,

$$\lim_{\varepsilon \to 0} \frac{2\varepsilon}{u^2 + \varepsilon^2} = 2\pi \delta(u).$$

We are therefore led from (5.51) to state

$$1^\wedge(u) = 2\pi \delta(u);$$

that is, the Fourier transform of 1 is $2\pi \delta(u)$. This confirms the result conjectured in Example 4, p. 27.

**Transforms of Distributions on the Line**

In attempting to define the Fourier transform of a distribution $t(x)$, we would like to use the formula (5.32), but unfortunately $e^{iu^2}x$ is not a test function in $K_1$, so that the action of $t$ on $e^{iu^2}$ is not defined. Instead we try to use Parseval's formula (5.35) to define $t^\wedge$ from

$$\langle t^\wedge, \varphi \rangle = \langle t, \varphi^\wedge \rangle.$$

Again the right side is not defined because $\varphi^\wedge$ is not a test function even though $\varphi$ is. The remedy is to introduce a more suitable class of test functions and correspondingly a new class of distributions. We begin with the one-dimensional case.

**Definition.** A complex-valued function $\varphi(x)$ of a single real variable is said to belong to $S_1$, the space of test functions of rapid decay, if

1. $\varphi(x)$ is infinitely differentiable.
2. $\varphi(x)$ together with all its derivatives vanish at $|x| = \infty$ faster than the reciprocal of any polynomial. Thus for each pair of nonnegative integers $k$ and $l$,

$$\lim_{|x| \to \infty} \left| x^k \frac{d^l \varphi}{dx^l} \right| = 0. \tag{5.52}$$

This class of test functions is larger than the class $K_1$ introduced in Section 5.2. The test functions in $K_1$ vanish identically outside a finite interval, whereas those in $S_1$ merely decrease rapidly at infinity. Every test function in $K_1$ also belong to $S_1$, but $e^{-x^2}$ belongs to $S_1$ and not to $K_1$. The test functions in $S_1$ form a linear space; moreover, if $\varphi$ is in $S_1$ so is $x^k \varphi^{(l)}(x)$ for any nonnegative integers $k$ and $l$.

**Convergence in $S_1$**

A sequence $\{\varphi_m(x)\}$ of functions in $S_1$ is said to be a *null sequence* in $S_1$ if for each pair of nonnegative integers $k$ and $l$,

$$\lim_{m \to \infty} \max_{-\infty < x < \infty} \left| x^k \frac{d^l \varphi_m}{dx^l} \right| = 0.$$
Definition. A distribution of slow growth is a continuous linear functional on $S_1$. Thus to each $\varphi$ in $S_1$ there is assigned a complex number $\langle t, \varphi \rangle$ with the properties

$$\langle t, \alpha_1 \varphi_1 + \alpha_2 \varphi_2 \rangle = \alpha_1 \langle t, \varphi_1 \rangle + \alpha_2 \langle t, \varphi_2 \rangle,$$

$$\lim_{m \to \infty} \langle t, \varphi_m \rangle = 0$$

for every null sequence in $S_1$.

Theorem. Every function $f(x)$ of slow growth generates a distribution of slow growth by the formula

$$\langle f, \varphi \rangle = \int_{-\infty}^{\infty} f(x) \varphi(x) dx,$$

Proof. The integral converges absolutely by the assumptions on $f$ and $\varphi$. It is also clear that the functional is linear; we must still prove continuity. Let $\varphi_n \to 0$ in $S_1$; then

$$\left| \int_{-\infty}^{\infty} f(x) \varphi_n(x) dx \right| = \left| \int_{-\infty}^{\infty} \frac{f(x)}{(1 + x^2)^p} (1 + x^2)^p \varphi_n(x) dx \right|.$$

For $p$ sufficiently large, $f(x)/(1 + x^2)^p$ is absolutely integrable from $-\infty$ to $\infty$, since $f$ is a function of slow growth. With such a value of $p$, the integral on the right is dominated by

$$\max_{-\infty < x < \infty} [(1 + x^2)^p |\varphi_n(x)|] \int_{-\infty}^{\infty} \frac{|f(x)|}{(1 + x^2)^p} dx.$$

Since $\varphi_n \to 0$ in $S_1$, the maximum which appears also approaches 0. Therefore, $\langle f, \varphi_n \rangle \to 0$ whenever $\varphi_n \to 0$ in $S_1$, and $\langle f, \varphi \rangle$ is a distribution of slow growth on $S_1$.

Nearly all important distributions on $K_1$ are also distributions on $S_1$. Only those distributions on $K_1$ which grow too rapidly at infinity cannot be extended to $S_1$. Much of the theory of Sections 5.1 to 5.5 can be applied to distributions on $S_1$ with but slight modifications. We shall accept this statement and proceed with the new aspects of the theory.

Theorem. If $\varphi$ is in $S_1$, then $\varphi^\wedge(u)$ exists and is also in $S_1$.

Proof. The rapid decay of $\varphi(x)$ at $|x| = \infty$ implies the absolute convergence of

$$\int_{-\infty}^{\infty} (ix)^k e^{iux} \varphi(x) dx, \quad k = 0, 1, 2, \ldots.$$
so that the quantity on the left is bounded for all \( u \). Moreover,

\[
(iu)^p \frac{d^k \phi}{du^k} = \int_{-\infty}^{\infty} (ix)^k \phi(x) \frac{d^p}{dx^p} e^{iux} \, dx,
\]

and by integration by parts the right side becomes

\[
(-1)^p \int_{-\infty}^{\infty} \left[ \frac{d^p}{dx^p} (ix)^k \phi(x) \right] e^{iux} \, dx.
\]

Since the term multiplying \( e^{iux} \) is in \( S_1 \), the integrand is absolutely integrable and therefore

\[
\left| u^p \frac{d^k \phi}{du^k} \right|
\]

is bounded for all \( u \). Since \( p \) and \( k \) are arbitrary, it follows that \( \phi^\wedge(u) \) is in \( S_1 \).

The same considerations also apply to the inverse transformation, so that we conclude that every function \( \psi(u) \) in \( S_1 \) is the transform of a function \( \phi(x) \) in \( S_1 \). Before proceeding with the principal task of defining the transform of a distribution of slow growth, we list some properties of the transforms of test functions in \( S_1 \). Consider the transform of \( d^k \phi/dx^k \); then

\[
\int_{-\infty}^{\infty} \frac{d^k \phi}{dx^k} e^{iux} \, dx = (-iu)^k \int_{-\infty}^{\infty} \phi e^{iux} \, dx,
\]

by integration by parts. In more compact notation,

\[
[\phi^{(k)}]^\wedge(u) = (-iu)^k \phi^\wedge(u). \tag{5.53}
\]

Also, we have

\[
\int_{-\infty}^{\infty} (ix)^k \phi(x) e^{iux} \, dx = \frac{d^k}{du^k} \int_{-\infty}^{\infty} \phi(x) e^{iux} \, dx;
\]

that is,

\[
[(ix)^k \phi]^\wedge(u) = \frac{d^k}{du^k} \phi^\wedge(u). \tag{5.54}
\]

By the inversion formula for Fourier transforms,

\[
\int_{-\infty}^{\infty} e^{ixz} \phi^\wedge(z) \, dz = 2\pi \phi(-x),
\]

and therefore

\[
\phi^\wedge^\wedge(x) = 2\pi \phi(-x). \tag{5.55}
\]

The change of variable \( x - a = y \) shows that

\[
\int_{-\infty}^{\infty} \phi(x - a) e^{iux} \, dx = e^{iua} \int_{-\infty}^{\infty} \phi(y) e^{iyv} \, dy,
\]
\[ [\varphi(x - a)]^\wedge(u) = e^{iau} \varphi^\wedge(u). \] (5.56)

**Definition.** Let \( t \) be any distribution of slow growth. Its Fourier transform \( t^\wedge \) always exists and is a distribution of slow growth defined from
\[ \langle t^\wedge, \varphi \rangle = \langle t, \varphi^\wedge \rangle. \] (5.57)

Since \( \varphi^\wedge \) is in \( S_1 \), the action of \( t \) on \( \varphi^\wedge \) is defined, so that \( t^\wedge \) is a functional on \( S_1 \). The functional is clearly linear; moreover, it is continuous, since whenever \( \varphi_m \to 0 \) in \( S_1 \), then \( \varphi_m^\wedge \to 0 \) in \( S_1 \) and therefore \( \langle t, \varphi_m^\wedge \rangle \to 0 \).

To show that the definition (5.57) really provides an extension of the usual definition of a Fourier transform, we must show that it coincides with the usual definition when \( t \) is an ordinary function which has a Fourier transform \( T \). Suppose this to be the case; then by (5.57),
\[ \langle t^\wedge, \varphi \rangle = \langle t, \varphi^\wedge \rangle = \int_{-\infty}^{\infty} t(x)dx \int_{-\infty}^{\infty} e^{iyx} \varphi(y)dy \]
\[ = \int_{-\infty}^{\infty} \varphi(y)dy \int_{-\infty}^{\infty} t(x)e^{ixy} dx \]
\[ = \int_{-\infty}^{\infty} T(y)\varphi(y)dy. \]

Thus the action of \( t^\wedge \) on \( \varphi \) is the same as the action of \( T \) on \( \varphi \). Hence \( t^\wedge = T \) and our definition is consistent.

We now show that the properties (5.53) to (5.56) hold for the transform \( t^\wedge \) of any distribution. First, consider the Fourier transform of the \( k \)th derivative of a distribution \( t \). Then, by the definition (5.57),
\[ \langle [t^{(k)}]^\wedge, \varphi \rangle = \langle t^{(k)}(u), \varphi^\wedge(u) \rangle = (-1)^k \langle t(u), (\varphi^\wedge)^{(k)}(u) \rangle. \]

Now by (5.54), we have
\[ (-1)^k \langle t, (\varphi^\wedge)^{(k)} \rangle = \langle t, [(-ix)^k \varphi]^\wedge \rangle = \langle t^\wedge(x), (-ix)^k \varphi(x) \rangle \]
\[ = \langle (-ix)^k t^\wedge(x), \varphi(x) \rangle, \]
where for the last step we have used the definition of multiplication of a distribution by an infinitely differentiable function of slow growth. Thus we find
\[ [t^{(k)}]^\wedge(x) = (-ix)^k t^\wedge(x) \] (5.58)
which is just (5.53) with a relabeling of the variables.

Turning next to the transform of \( (ix)^k t \),
\[ \langle [(ix)^k t]^\wedge, \varphi \rangle = \langle (ix)^k t(x), \varphi^\wedge(x) \rangle = (-1)^k \langle t(x), (-ix)^k \varphi^\wedge(x) \rangle, \]
and, by using (5.53),
\[ \langle t(x), (-ix)^k \varphi^\wedge(x) \rangle = \langle t(x), (\varphi^{(k)})^\wedge(x) \rangle = \langle t^\wedge(u), \varphi^{(k)}(u) \rangle \]
\[ = (-1)^k \left\langle \frac{d^{k^\wedge}}{du^k} (u), \varphi(u) \right\rangle. \]
Consequently,

\[ [(ix)^k] \hat{\varphi}(u) = \frac{d^k \varphi}{du^k}. \] (5.59)

We leave the proofs of the following properties to the reader.

\[ t^\wedge(x) = 2\pi t(-x), \] (5.60)

\[ [t(x - a)]^\wedge(u) = e^{i\omega u} t^\wedge(u), \] (5.61)

\[ [t(-x)]^\wedge(u) = t^\wedge(-u). \] (5.62)

**Examples**

**Example 1.** Consider the transform of \(\delta(x)\). Then

\[
\langle \delta^\wedge, \varphi \rangle = \langle \delta(x), \varphi^\wedge(x) \rangle = \left\langle \delta(x), \int_{-\infty}^{\infty} \varphi(y)e^{ixy} \, dy \right\rangle 
\]

\[
= \int_{-\infty}^{\infty} \varphi(y) \, dy = \langle 1, \varphi \rangle.
\]

Therefore,

\[ \delta^\wedge = 1. \] (5.63)

**Example 2.** To find the transform of \(f(x) = 1\),

\[
\langle 1^\wedge, \varphi \rangle = \langle 1, \varphi^\wedge(x) \rangle = \int_{-\infty}^{\infty} \varphi(x) \, dx 
\]

\[
= \left[ \int_{-\infty}^{\infty} \varphi(x)e^{ixy} \, dx \right]_{y=0}.
\]

By the inversion formula for \(\varphi^\wedge\), this last integral is just \(2\pi \varphi(0)\). Thus

\[ 1^\wedge = 2\pi \delta. \] (5.64)

The same result can also be obtained by using (5.63) and (5.60). In fact, from (5.63),

\[ 1^\wedge = \delta^\wedge \wedge, \]

and, from (5.60),

\[ \delta^\wedge(x) = 2\pi \delta(-x) = 2\pi \delta(x). \]

**Example 3.** The transform of \(\delta(x - a)\) is \(e^{i\omega u}\).

**Example 4.** We now calculate the transform of the Heaviside function \(H(x)\) in a number of different ways.

(a)

\[
\langle H^\wedge, \varphi \rangle = \langle H, \varphi^\wedge \rangle = \int_{0}^{\infty} \varphi^\wedge(x) \, dx 
\]

\[
= \int_{0}^{\infty} \, dx \int_{-\infty}^{\infty} \varphi(y)e^{ixy} \, dy.
\]
\[ \lim_{R \to \infty} \int_{-\infty}^{\infty} \varphi(y) dy \int_{0}^{R} e^{i xy} \, dx \]
\[ = \lim_{R \to \infty} \int_{-\infty}^{\infty} \frac{e^{i Ry} - 1}{iy} \varphi(y) dy. \]

Now by Exercise 1.26, \( \lim_{R \to \infty} (1 - \cos Ry)/y = \text{pf}(1/y) \). Moreover \( \lim_{R \to \infty} \sin Ry/y = \pi \delta(y) \). Therefore,

\[ \langle H^\wedge, \varphi \rangle = \left\langle \text{ipf} \frac{1}{y} + \pi \delta(y), \varphi(y) \right\rangle, \]
\[ H^\wedge(y) = \text{ipf} \frac{1}{y} + \pi \delta(y). \quad (5.65) \]

(b) We have \( H'(x) = \delta(x) \). By (5.58),

\[ (H')^\wedge(x) = -ixH^\wedge(x), \]
and, using (5.63),

\[ 1 = -ixH^\wedge(x). \]

Thus \( H^\wedge \) satisfies the distributional equation

\[ 1 = -ixt(x), \quad (5.66) \]

a particular solution of which is

\[ t(x) = \text{ipf} \frac{1}{x}. \]

In fact, substituting in (5.66) we find

\[ \langle 1, \varphi \rangle = \left\langle \text{xfp} \frac{1}{x}, \varphi \right\rangle = \left\langle \text{pf} \frac{1}{x}, x\varphi \right\rangle \]
\[ = \lim_{\varepsilon \to 0} \int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty} \left( \frac{1}{x} \right) x\varphi \, dx = \int_{-\infty}^{\infty} \varphi \, dx, \]

which is an identity.

To find the general solution of (5.66) we must add to the particular solution just obtained the general solution of the homogeneous equation \(-ixt = 0\), which by Exercise 1.31, is \( C \delta(x) \). Therefore,

\[ H^\wedge(x) = \text{ipf} \frac{1}{x} + C \delta(x), \quad (5.67) \]

where \( C \) is a constant to be determined. The following trick enables us to find \( C \). Consider the equation

\[ H(x) + H(-x) = 1, \]
whose transform by (5.62) and (5.64) yields

\[ H^\wedge(u) + H^\wedge(-u) = \delta(u). \]

From (5.67), we then find \( 2C = 2\pi \) or \( C = \pi \). Thus

\[ H^\wedge(x) = \text{ipf}\, \frac{1}{x} + \pi \delta(x). \]

\textbf{(c)} The distribution \( H(x) \) may be considered as the limit as \( \varepsilon \to 0^+ \) of \( H(x)e^{-\varepsilon x} \) (see Exercise 5.15). Therefore, by Exercise 5.16, \( H^\wedge = \lim_{\varepsilon \to 0^+} (H e^{-\varepsilon x})^\wedge. \)

But \( H e^{-\varepsilon x} \) has a Fourier transform in the ordinary sense:

\[ (H e^{-\varepsilon x})^\wedge = \int_0^\infty e^{-\varepsilon x} e^{iux} \, dx = \frac{1}{\varepsilon - iu} = \frac{\varepsilon + iu}{\varepsilon^2 + u^2}. \]

The distributional limit as \( \varepsilon \to 0^+ \) is easily calculated. We have

\[ \lim_{\varepsilon \to 0^+} \frac{\varepsilon}{\varepsilon^2 + u^2} = \pi \delta(u), \]

\[ \lim_{\varepsilon \to 0^+} \frac{iu}{\varepsilon^2 + u^2} = \text{ipf}\, \frac{1}{u}, \]

which leads again to (5.65).

\section*{Transforms in More Than One Variable}

\textbf{Definition.} A complex-valued function \( \varphi(x_1, \ldots, x_n) = \varphi(x) \) is said to belong to \( S_n \), the space of \( n \)-dimensional test functions of rapid decay, if

1. \( \varphi(x) \) is infinitely differentiable; that is \( D^l \varphi \) exists for any multiindex \( l \) of order \( n \).

2. For each pair of multiindices \( k \) and \( l \) of order \( n \),

\[ \lim_{|x| \to \infty} |x^k D^l \varphi| = 0, \]

where

\[ x^k = x_1^{k_1} \cdots x_n^{k_n}, \]

\[ D^l \varphi = \frac{\partial^{l_1 + \cdots + l_n}}{\partial x_1^{l_1} \cdots \partial x_n^{l_n}}. \]

A sequence \( \{ \varphi_m(x) \} \) of functions in \( S_n \) is said to be a null sequence in \( S_n \) if for each pair of multiindices \( k \) and \( l \) of order \( n \),

\[ \lim_{m \to \infty} \max_{x \in R_n} |x^k D^l \varphi_m| = 0. \]
DEFINITION. An \( n \)-dimensional distribution of slow growth is a continuous linear functional on \( S_n \). To each test function \( \varphi \) in \( S_n \) there is assigned a complex number \( \langle t, \varphi \rangle \) with the properties

\[
\langle t, \alpha_1 \varphi_1 + \alpha_2 \varphi_2 \rangle = \alpha_1 \langle t, \varphi_1 \rangle + \alpha_2 \langle t, \varphi_2 \rangle
\]

\[
\lim_{m \to \infty} \langle t, \varphi_m \rangle = 0 \quad \text{for every null sequence in } S_n.
\]

We can now define \( n \)-dimensional transforms. First, if \( \varphi \) is in \( S_n \), we define

\[
\varphi^\wedge(u) = \varphi^\wedge(u_1, \ldots, u_n) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \varphi(x_1, \ldots, x_n) e^{iu_1 x_1} \cdots e^{iu_n x_n} \, dx_1 \cdots dx_n
\]

\[
= \int_{R_n} \varphi(x) e^{iu \cdot x} \, dx.
\]

It is easily seen that \( \varphi^\wedge(u) \) is an \( n \)-dimensional test function of rapid decay. The inversion formula is

\[
\varphi(x) = \frac{1}{(2\pi)^n} \int_{R_n} \varphi^\wedge(u) e^{-iu \cdot x} \, du.
\]

The transform of a distribution of slow growth is then defined by use of Parseval's formula

\[
\langle t^\wedge, \varphi \rangle = \langle t, \varphi^\wedge \rangle.
\]

As examples we observe that

\[
1^\wedge = (2\pi)^n \delta,
\]

\[
\delta^\wedge = 1,
\]

where \( 1 \) is the constant function equal to 1 everywhere in \( R_n \) and \( \delta \) is the \( n \)-dimensional Dirac distribution. Formulas similar to (5.58) to (5.62) are easily derived and are left to the reader.

EXERCISES

5.15 Show that if \( f(x) \) is a function of slow growth on the real line,

\[
\lim_{\varepsilon \to 0^+} \langle f(x) e^{-\varepsilon |x|}, \varphi(x) \rangle = \langle f, \varphi \rangle.
\]

Thus in the distributional sense

\[
\lim_{\varepsilon \to 0^+} f(x) e^{-\varepsilon |x|} = f(x).
\]

5.16 Let \( t_n \) be a sequence of distributions on \( S_1 \) such that \( t_n \to t \), where \( t \) is a distribution on \( S_1 \). Show that \( t_n^\wedge \to t^\wedge \).

5.17 Let \( f(x) \) be a function of slow growth on the real line; then by Exercise 5.15,

\[
f(x) = \lim_{\varepsilon \to 0^+} f(x) e^{-\varepsilon |x|} = \lim_{\varepsilon \to 0^+} [f_+(x) e^{-\varepsilon x} + f_-(x) e^{\varepsilon x}].
\]
Therefore, by Exercise 5.16,
\[ f^\wedge(u) = \lim_{\varepsilon \to 0^+} \left[ \int_{-\infty}^{\infty} f(x)e^{-\varepsilon x}e^{ux} \, dx + \int_{0}^{\infty} f(x)e^{\varepsilon x}e^{ux} \, dx \right] \]

or
\[ f^\wedge(u) = \lim_{\varepsilon \to 0^+} [f^\wedge(u + i\varepsilon) + f^\wedge(u - i\varepsilon)], \]

where \( f^\wedge \) and \( f^\wedge \) are the one-sided transforms defined in (5.46) and (5.47). Thus we have established (5.51). In particular, if \( f \) vanishes for \( x < 0 \), we have
\[ f^\wedge(u) = \lim_{\varepsilon \to 0^+} f^\wedge(u + i\varepsilon). \]

5.18 Let \( t \) be a distribution of slow growth on \( S_1 \). It can be shown that there exists an integer \( p \) such that
\[ t = \frac{d^pf}{dx^p}, \]

where \( f \) is a function of slow growth. Show that
\[ t^\wedge(u) = (-iu)^p \lim_{\varepsilon \to 0^+} [f^\wedge(u + i\varepsilon) + f^\wedge(u - i\varepsilon)]. \]

5.19 Let \( f(x) \) be a right-sided function of slow growth on the real line. Its **Laplace transform** is defined by
\[ \mathcal{F}(s) = \int_{0}^{\infty} e^{-sx}f(x)dx, \]

where \( s \) is a complex variable. Clearly the integral converges and represents an analytic function for the real part of \( s \) positive. In fact, the definition is exactly that of the right-sided Fourier transform (5.46), where we have set \( s = -i\omega \). According to (5.48), we have the inversion formula
\[ f(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{sx}\mathcal{F}(s)ds, \]

where \( a > 0 \). Thus the inversion integral is taken along any vertical line in the right half-plane and the integral vanishes for \( x < 0 \).

5.20 Find the Fourier transforms of the following distributions on \( S_1 \):

(a) \( \text{pf} \frac{1}{x} \).

(b) \( \text{sgn } x = \begin{cases} 1, & x > 0; \\ -1, & x < 0. \end{cases} \)

(c) \( \log |x| \).

It will help to recall that \( (d/dx) \log |x| = \text{pf} (1/x) \).
5.21 Find the Fourier transform of \( x^k \), where \( k \) is a positive integer and \( x \) is a single real variable.

5.22 Let \( t \) be the distribution corresponding to a simple layer of surface density \( a(x) \) on the closed, bounded hypersurface \( \sigma \) in \( R_n \). We calculate \( t^\wedge \) by the following formal calculation:

\[
\langle t^\wedge, \varphi \rangle = \langle t, \varphi^\wedge \rangle = \int_\sigma a(x)\varphi^\wedge(x) dS_x
\]

\[
= \int_\sigma a(x) dS_x \int_{R_n} \varphi(\xi) e^{i\xi \cdot x} d\xi = \int_{R_n} \varphi(\xi) d\xi \int_\sigma a(x) e^{i\xi \cdot x} dS_x.
\]

Therefore,

\[
t^\wedge(\xi) = \int_\sigma a(x) e^{i\xi \cdot x} dS_x. \tag{5.68}
\]

Now suppose \( \sigma \) is the surface of a sphere of radius \( R \) and that \( a(x) = 1 \) (a simple layer of unit surface density). Then \( t(x) = \delta(r - R) \), where \( r \) is the radial spherical coordinate. Passing to spherical coordinates in (5.68), we find

\[
t^\wedge(\xi) = R^{n-1} S_{n-1}(1) \int_0^\infty e^{iR|\xi|\cos\theta} \sin^{n-2} \theta d\theta, \tag{5.69}
\]

where \( S_{n-1}(1) \) is the surface area of the unit sphere in \( n - 1 \) dimensions. As expected, \( t^\wedge \) is spherically symmetric in the transform variable \( \xi \).

In the cases \( n = 2 \) and \( n = 3 \), obtain the formulas

\[
t^\wedge(\xi) = 2\pi RJ_0(R|\xi|), \tag{5.70}
\]

\[
t^\wedge(\xi) = 4\pi R \frac{\sin R|\xi|}{|\xi|}, \tag{5.71}
\]

respectively.

5.7 PARTIAL DIFFERENTIAL EQUATIONS FOR DISTRIBUTIONS

**Green's Theorem**

Let \( L \) be an arbitrary linear differential operator of order \( p \) in the \( n \) independent variables \( x_1, \ldots, x_n \). In the compact notation (5.1), we have

\[
L = \sum_{|k| \leq p} a_k(x) D^k,
\]

where we shall assume that the functions \( a_k(x) \) have partial derivatives of all orders. The formal adjoint of \( L \) is denoted by \( L^* \); it was defined in (5.12) and we repeat the definition for convenience:

\[
L^*v = \sum_{|k| \leq p} (-1)^{|k|} D^k(a_k v).
\]
Now let $u$ and $v$ be functions having continuous derivatives of order $p$ in $R_n$; that is, $D^k u$ and $D^k v$ are continuous for every multiindex $k$ with $|k| \leq p$. Then it can be shown that

$$v Lu - u L^* v = \text{div} \, J(u, v), \quad (5.72)$$

where $J$ is a vectorial bilinear form in $u$ and $v$ involving only derivatives of $u$ and $v$ of order $p - 1$ or less. Equation (5.72) is a generalization of the one-dimensional version (for second-order operators) derived in Equation (1.66). The integral form of (5.72) is known as Green's theorem:

$$\int_R (v Lu - u L^* v) dx = \int_\sigma n \cdot J \, dS. \quad (5.73)$$

Here $R$ is a bounded region in $R_n$ with bounding surface $\sigma$ whose exterior normal is denoted by $n$. We give a few illustrations.

**EXAMPLES**

**Example 1.** Consider the Laplace operator

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2},$$

which is formally self-adjoint; that is, $(\nabla^2)^* = \nabla^2$. Equation (5.72) becomes

$$v \nabla^2 u - u \nabla^2 v = \text{div} (v \, \text{grad} \, u - u \, \text{grad} \, v), \quad (5.74)$$

and Green's theorem takes the usual form

$$\int_R (v \nabla^2 u - u \nabla^2 v) dx = \int_\sigma \left( v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) dS. \quad (5.75)$$

**Example 2.** For the biharmonic operator,

$$\nabla^4 = \nabla^2 \nabla^2 = \left( \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right) \left( \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right),$$

we easily establish the formal self-adjointness

$$(\nabla^4)^* = \nabla^4.$$

Equation (5.72) becomes

$$v \nabla^4 u - u \nabla^4 v = \text{div} \, J(u, v),$$

where

$$J(u, v) = v \, \text{grad} \, \nabla^2 u - u \, \text{grad} \, \nabla^2 v + (\nabla^2 v) \, \text{grad} \, u - (\nabla^2 u) \, \text{grad} \, v.$$

Green's theorem has the form

$$\int_R (v \nabla^4 u - u \nabla^4 v) dx = \int_\sigma \left[ v \frac{\partial \nabla^2 u}{\partial n} - u \frac{\partial \nabla^2 v}{\partial n} + (\nabla^2 v) \frac{\partial u}{\partial n} - (\nabla^2 u) \frac{\partial v}{\partial n} \right] dS. \quad (5.76)$$
Example 3. The heat-conduction operator is
\[
L = \frac{\partial}{\partial t} - \left( \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right),
\]
(5.77)
where the time variable \( t \) has been distinguished from the space variables \( x_1, \ldots, x_n \). We easily calculate
\[
L^* = -\frac{\partial}{\partial t} - \left( \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right).
\]
(5.78)
Then
\[
vLu - uL^*v = \text{div} \, J,
\]
where
\[
J = e_t u v - (v \text{ grad}_x u - u \text{ grad}_x v),
\]
the subscript \( x \) indicating differentiation with respect to the space variables alone; \( e_t \) is a unit vector in the time direction.

For Green's theorem, we find
\[
\int_R (vLu - uL^*v)dx \, dt = \int_\sigma n \cdot [e_t u v + u \text{ grad}_x v - v \text{ grad}_x u]dS,
\]
(5.79)
where \( R \) is a bounded region in space-time, \( \sigma \) is its boundary, and \( dx \, dt \) is an element of volume in space-time.

Example 4. Consider the wave operator
\[
\Box^2 = \frac{\partial^2}{\partial t^2} - \left( \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right),
\]
(5.80)
which is clearly formally self-adjoint. Then we find
\[
v\Box^2u - u\Box^2v = \text{div} \left[ e_t \left( v \frac{\partial u}{\partial t} - u \frac{\partial v}{\partial t} \right) - v \text{ grad}_x u + u \text{ grad}_x v \right],
\]
with the same terminology as in Example 3. Green's theorem becomes
\[
\int_R (v\Box^2u - u\Box^2v)dx \, dt = \int_\sigma n \cdot \left[ e_t \left( v \frac{\partial u}{\partial t} - u \frac{\partial v}{\partial t} \right) + u \text{ grad}_x v - v \text{ grad}_x u \right]dS.
\]
(5.81)
The unit vectors in the space directions are denoted by \( e_1, \ldots, e_n \), whereas the unit vector in the time direction is \( e_t \). We introduce a vector \( q \), known as the transversal to \( \sigma \), defined by
\[
q \cdot e_t = n \cdot e_t,
q \cdot e_1 = -n \cdot e_1, \ldots, q \cdot e_n = -n \cdot e_n.
\]
(5.82)
Clearly \( q \) is a unit vector which defines a particular direction at each point on \( \sigma \). In terms of \( q \), (5.81) simplifies to

\[
\int_R \left( v \Box^2 u - u \Box^2 v \right) dx \; dt = \int_\sigma \left( v \frac{\partial u}{\partial q} - u \frac{\partial v}{\partial q} \right) dS,
\]

(5.83)

where \( \partial/\partial q \) is the derivative in the transversal direction. When dealing with the wave operator in one space dimension, the boundary of the two-dimensional region \( R \) in space-time is a closed curve \( C \). If \( dl \) denotes an element of length on \( C \), (5.83) becomes

\[
\int_R \left[ v \left( \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} \right) - u \left( \frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial x^2} \right) \right] dx \; dt = \int_C \left( v \frac{\partial u}{\partial q} - u \frac{\partial v}{\partial q} \right) dl.
\]

(5.83a)

**Strict and Generalized Solutions of Differential Equations**

Let \( s(x) \) be a given continuous function in \( R_n \), and consider the partial differential equation

\[
Lu = s.
\]

(5.84)

A function \( u(x) \) is said to be a *strict solution* of (5.84) in an open region \( R \) in \( R_n \), if \( u \) has continuous derivatives of order \( p \) in \( R \) and if at every point of \( R \) we have \( Lu = s \).

If \( s \) is a given distribution (and this, of course, includes the possibility that \( s \) is a continuous function), we may consider (5.84) as an equation for an unknown distribution \( u \). We shall say that \( u \) is a *generalized solution* of (5.84) if it satisfies the equation in the distributional sense, that is, if \( \langle Lu, \phi \rangle = \langle s, \phi \rangle \). Here the left side is defined for any distribution \( u \) from \( \langle Lu, \phi \rangle = \langle u, L^*\phi \rangle \), so that \( u \) is a generalized solution of (5.84) if and only if

\[
\langle u, L^*\phi \rangle = \langle s, \phi \rangle,
\]

(5.85)

for every test function \( \phi \) in \( K_n \).

Although (5.85) gives no hint for finding a solution \( u \), it enables us in principle to determine if a distribution \( u \) actually satisfies (5.84). We only have to verify that, for each \( \phi \), the action of \( u \) on the test function \( L^*\phi \) is the same as the action of \( s \) on \( \phi \).

The definition just given for a generalized solution is global in character; that is, it applies to the whole of \( R_n \). We would also like to define a notion of generalized solution in an open region \( R \). In the light of our earlier discussion of the values of a distribution, it is natural to use the following definition: \( u \) is a *generalized solution* of (5.84) in \( R \) if (5.85) holds for all test functions \( \phi \) with support contained in \( R \).

**Homogeneous Equation**

As a special case of (5.84) we investigate the homogeneous equation

\[
Lu = 0.
\]

(5.86)
A distribution $u$ is a generalized solution of (5.86) in the open region $R$ if and only if

$$\langle u, L^* \varphi \rangle = 0,$$

for every test function $\varphi$ whose support is contained in $R$.

**Theorem 1.** Every strict solution of (5.86) in $R$ is also a generalized solution in $R$.

**Proof.** Let $u$ be a strict solution in $R$. Then $u$ is a function with continuous derivatives of order $p$; thus if the support of $\varphi$ is contained in $R$, we have

$$\langle u, L^* \varphi \rangle = \int_R u L^* \varphi \, dx.$$

We can apply Green's theorem (5.73) and, by observing that $\varphi$ vanishes in a neighborhood of the boundary $\sigma$ of $R$, we conclude that $J$ vanishes on $\sigma$. Thus

$$\langle u, L^* \varphi \rangle = \int_R \varphi Lu \, dx,$$

and, since $Lu = 0$ at every point of $R$,

$$\langle u, L^* \varphi \rangle = 0.$$

**Theorem 2.** Let $u$ be a function with continuous derivatives of order $p$ which is a generalized solution of (5.86) in $R$; then $u$ is a strict solution in $R$.

**Proof.** By assumption,

$$\langle u, L^* \varphi \rangle = 0,$$

for every $\varphi$ with support contained in $R$. By the same argument as in the preceding proof,

$$\langle u, L^* \varphi \rangle = \int_R \varphi Lu \, dx,$$

where $Lu$ is a continuous function, say, $Lu = q$. We therefore know that for every $\varphi$ with support contained in $R$,

$$\int_R \varphi q \, dx = 0,$$

and we wish to show that this implies $q \equiv 0$ in $R$. Suppose $q(x_0)$ were not 0, say, $q(x_0) > 0$. We can then find a sphere with center at $x_0$, lying wholly in $R$, and such that $q > 0$ in the interior of the sphere. We can also construct a test function $\varphi$ positive inside this sphere and vanishing elsewhere. For this $\varphi$ we would have $\int_R \varphi q \, dx > 0$, contradicting the hypothesis. Therefore, $q = Lu = 0$ at every point in $R$ and $u$ is a strict solution in $R$.  

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The question arises as to whether (5.86) can have generalized solutions which are not strict solutions. We have seen in Chapter I that this can happen for ordinary differential equations only if the coefficient $a_p(x)$ of the derivative of the highest order appearing in $L$ vanishes at some point in the open interval $R$. The situation is more complicated for partial differential equations. The role formerly played by the coefficient of the highest-order term is now taken over by a matrix of the coefficients of the terms of order $p$. We adopt this point of view in Section 5.9. For the present we content ourselves with some examples.

Consider the homogeneous wave equation in one space dimension $x$:

$$Lu = \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0. \quad (5.87)$$

If $f(y)$ is any function of a real variable $y$ with a continuous second derivative, then

$$u(x, t) = f(x - t) \quad (5.88)$$

is easily seen to be a strict solution of (5.87). Indeed

$$\frac{\partial^2 u}{\partial x^2} = f''(x - t), \quad \frac{\partial^2 u}{\partial t^2} = f''(x - t),$$

so that $Lu = 0$.

The physical interpretation of (5.88) is simple and quite revealing. The solution (5.88) represents a wave traveling to the right with unit velocity; that is, at times $t_1 < t_2$, we have the same wave form for the solution, the one at time $t_2$ being displaced by an amount $t_2 - t_1$ to the right from the one at time $t_1$.

There seems to be no reason physically to restrict oneself to wave forms which have a continuous second derivative. This suggests that even if $f$ is not differentiable, $u$ as given by (5.88) might still be a generalized solution of (5.87). Consider, for instance, the case where $f$ is the Heaviside function; then

$$u(x, t) = \begin{cases} 1, & x > t; \\ 0, & x < t, \end{cases}$$

and we proceed to show that $u$ is a generalized solution of (5.87). Since $L^* = L$, we have

$$\langle u, L^* \varphi \rangle = \iint_{x > t} \left( \frac{\partial^2 \varphi}{\partial t^2} - \frac{\partial^2 \varphi}{\partial x^2} \right) dx \, dt. \quad (5.89)$$

Being a test function, $\varphi$ vanishes outside a bounded region $Q$; hence the integration in (5.89) takes place over the bounded region $R$ which is the intersection of the regions $Q$ and $x > t$. By using (5.83a), we find

$$\langle u, L^* \varphi \rangle = \iint_{x > t} \varphi \left( \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} \right) dx \, dt + \int_{x = t} \frac{\partial \varphi}{\partial q} dl,$$
where we have used the fact that \( \varphi \) and all its derivatives vanish on the remainder of the boundary of \( R \). Now from (5.82) we see that the transversal \( q \) to the line \( x = t \) is along the line \( x = t \); hence \( \partial \varphi / \partial t = \partial \varphi / \partial l \). Since \( \varphi \) vanishes outside a finite segment of the line \( x = t \), we have
\[
\int_{x=t} \frac{\partial \varphi}{\partial l} \, dl = 0.
\]
Moreover, \((\partial^2 u/\partial t^2) - (\partial^2 u/\partial x^2) = 0\) in the strict sense for \( x > t \), so that
\[
\langle u, L^* \varphi \rangle = 0
\]
and \( u \) is a generalized solution of (5.87).

Now that we have established that a discontinuous function can be a generalized solution of the wave equation (5.87) we would like to characterize all such solutions \( u(x, t) \). Let the curve \( C \) separate the \( xt \) plane into the two parts \( Q_- \) and \( Q_+ \); in \( Q_- \) and \( Q_+ \), \( u(x, t) \) has continuous derivatives of the second order and, moreover, \( u \) and its first derivatives have limits as we approach \( C \) from either side, but we do not assume that the limits from either side are equal. Thus \( u \) and its first derivatives may have simple discontinuities on \( C \). If \( u \) is to be a generalized solution in the plane it is certainly a generalized solution in \( Q_- \) and \( Q_+ \), and therefore, by Theorem 2, \( u \) is a strict solution in both \( Q_- \) and \( Q_+ \). From the definition of a generalized solution, we have
\[
0 = \langle u, L^* \varphi \rangle = \iint_{Q_2} u \left( \frac{\partial^2 \varphi}{\partial t^2} - \frac{\partial^2 \varphi}{\partial x^2} \right) \, dx \, dt
\]
\[
= \iint_{Q_-} u \left( \frac{\partial^2 \varphi}{\partial t^2} - \frac{\partial^2 \varphi}{\partial x^2} \right) \, dx \, dt + \iint_{Q_+} u \left( \frac{\partial^2 \varphi}{\partial t^2} - \frac{\partial^2 \varphi}{\partial x^2} \right) \, dx \, dt.
\]

Using Green's theorem (5.83a) and the fact that \( \varphi \) vanishes outside a bounded region, we can reduce each double integral to a line integral on \( C \). From (5.82) we see that the transversals have opposite directions depending on whether we regard \( C \) as the boundary of \( Q_+ \) or \( Q_- \). Let \( n \) be the outward normal to \( Q_+ \) and let \( q \) be the transversal formed by means of this normal. Then
\[
0 = \int_C \left[ (\Delta u) \frac{\partial \varphi}{\partial q} - \varphi \left( \Delta \frac{\partial u}{\partial q} \right) \right] \, dl,
\]
(5.90)
where the prefix \( \Delta \) indicates that the jump across \( C \) of the quantity following it is to be taken. Two distinct cases must be considered. First, suppose that the transversal \( q \) to \( C \) is not tangent to \( C \); then \( \varphi \) and \( \partial \varphi / \partial q \) are independent and the integral (5.90) will vanish for all \( \varphi \) if and only if
\[
\Delta u = 0 \quad \text{and} \quad \Delta \frac{\partial u}{\partial q} = 0.
\]
Thus if the transversal is not tangent to $C$, $C$ cannot be a locus of discontinuity for the generalized solution $u$ or any of its first derivatives. On the other hand, suppose that $q$ is tangent to $C$; that is, $q \cdot n = 0$. From (5.82),

$$q = (n \cdot e_t)e_t - (n \cdot e_i)e_i,$$

where $e_1$ and $e_i$ are unit vectors in the $x$ and $t$ directions, respectively. We have

$$0 = q \cdot n = (n \cdot e_t)^2 - (n \cdot e_i)^2$$

or

$$n \cdot e_t = \pm n \cdot e_i.$$

Hence $q$ is tangent to $C$ if and only if $C$ is one of families of straight lines $x - t = \text{constant}$ or $x + t = \text{constant}$. These curves are called characteristics (see Section 5.9 for a more detailed discussion). If $C$ is a characteristic, then $\partial/\partial q$ is proportional to $\partial/\partial l$ and (5.90) becomes

$$0 = \int_C \left[ (\Delta u) \frac{\partial \varphi}{\partial l} - \varphi \left( \Delta \frac{\partial u}{\partial l} \right) \right] dl.$$

Now $\Delta(\partial u/\partial l) = (\partial/\partial l)(\Delta u)$, and we can rewrite the preceding equation as

$$0 = \int_C \left[ \frac{\partial}{\partial l} (\varphi \Delta u) - 2 \varphi \frac{\partial}{\partial l} (\Delta u) \right] dl. \quad (5.91)$$

Since $\varphi$ vanishes outside a finite segment of $C$, the first integral, which equals the difference in $\varphi \Delta u$ at the upper and lower limits, must vanish. Therefore,

$$0 = \int_C \varphi \frac{\partial}{\partial l} (\Delta u) dl,$$

and because $\varphi$ is an arbitrary test function, we can infer

$$\frac{\partial}{\partial l} \Delta u = 0 \quad \text{or} \quad \Delta u = \text{constant on } C. \quad (5.92)$$

Thus we have shown that the only curves which can carry discontinuities are the characteristics, and that across a characteristic the jump in the solution remains constant. Exercise 5.26 shows that characteristics can propagate even stronger singularities than the ones considered above.

Next we want to investigate the possibility that the $n$-dimensional Laplace equation might have generalized solutions which have a discontinuity on a hypersurface $\sigma$ which separates $R_n$ into the two regions $Q_+$ and $Q_-$. If the generalized solution $u$ and its first derivatives have simple discontinuities on $\sigma$,

$$0 = \langle u, \nabla^2 \varphi \rangle = \int_{R_n} u \nabla^2 \varphi \, dx$$

$$= \int_{Q_-} u \nabla^2 \varphi \, dx + \int_{Q_+} u \nabla^2 \varphi \, dx.$$
We appeal to Green's theorem (5.75) in each of the regions $Q_-$ and $Q_+$ separately to obtain

$$0 = \int_\sigma \left[ (\Delta u) \frac{\partial \varphi}{\partial n} - \left( \Delta \frac{\partial u}{\partial n} \right) \varphi \right] dS,$$

where $n$ is the outward normal to $\sigma$ from $Q_+$, and where we have used the fact that $u$ is a strict solution in the regions $Q_-$ and $Q_+$.

Since $\varphi$ and $\partial \varphi / \partial n$ are independent, we must have $\Delta u = 0$ and $\Delta (\partial u / \partial n) = 0$. The first of these relations also implies that all tangential derivatives of $u$ are continuous on crossing $\sigma$. Thus $u$ and all its first derivatives are continuous on $\sigma$, and Laplace's equation does not have any generalized solutions with simple discontinuities in either $u$ or its first derivatives. In fact, it can be shown that every generalized solution of Laplace's equation is a strict solution and that, moreover, such strict solutions are infinitely differentiable. This last remark is borne out by our experience with the two-dimensional case, where we know that any solution is the real part of an analytic function of the complex variable $x_1 + ix_2$.

**Exercises**

5.23 In most applications the wave operator appears in the form

$$L = \frac{\partial^2}{\partial t^2} - c^2 \left( \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right), \quad (5.93)$$

where $c^2$ is a positive constant, instead of in the form (5.80). Of course, the change of variable $\tau = ct$ reduces (5.93) to (5.80), but here we shall consider (5.93) directly.

Write Green's theorem for (5.93). Assuming that the piecewise strict generalized solution $u$ has a simple discontinuity on $\sigma$, show that $\sigma$ satisfies

$$c^2 \sum_{k=1}^{n} (n \cdot e_k)^2 = (n \cdot e_\tau)^2. \quad (5.94)$$

5.24 Consider the first-order equation

$$\frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} = 0.$$

Show that the only characteristics are the family of straight lines $x_1 - x_2 = \text{constant}$. Verify that $u = H(x_1 - x_2)$ is a generalized solution of the equation.

5.25 Show that

$$\nabla^4 u = 0$$

cannot have piecewise strict generalized solutions with simple discontinuities in $D^k u$ on a hypersurface $\sigma$ (restrict yourself to the case $|k| \leq 3$).
5.26 Show that \( \delta(x - t) \) and \( \delta(x + t) \) are generalized solutions of (5.87).

*Hint:* If \( \varphi(x, t) \) is a two-dimensional test function

\[
\langle \delta(x - t), \varphi(x, t) \rangle = \int_{-\infty}^{\infty} \varphi(x, \alpha)d\alpha.
\]

Therefore you must prove that for every \( \varphi \)

\[
\int_{-\infty}^{\infty} \psi(\alpha, \alpha)d\alpha = 0,
\]

where

\[
\psi(x, t) = \frac{\partial^2 \varphi}{\partial t^2} - \frac{\partial^2 \varphi}{\partial x^2};
\]

this is easily done by introducing new variables \( x - t \) and \( x + t \).

5.27 Show that \( \delta(x_1 - a) \) and \( \delta(x_2 - b) \) are generalized solutions of

\[
\frac{\partial^2 u}{\partial x_1 \partial x_2} = 0.
\]

5.28 Show that every piecewise strict generalized solution of

\[
\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0
\]

is a strict solution.

5.8 FUNDAMENTAL SOLUTIONS

A powerful method for the study of linear partial differential equations makes use of the *fundamental solution* (see Chapter 1 for the one-dimensional treatment). Let \( L \) be a linear partial differential operator of order \( p \), given in general form by (5.1). A fundamental solution for \( L \) with pole at \( \xi \) is a distribution \( E \) in \( x \) (depending parametrically on \( \xi \)), denoted by \( E(x | \xi) \), which satisfies

\[
LE = \delta(x - \xi).
\] (5.95)

Thus \( E \) is the response to a concentrated unit source located at \( x = \xi \). By definition a distribution satisfies (5.95) if and only if \( \langle LE, \varphi \rangle = \varphi(\xi) \) for each test function \( \varphi \). Since \( E \) cannot be sufficiently differentiable for \( LE \) to make sense classically, we understand that \( \langle LE, \varphi \rangle \) is defined distributionally from \( \langle LE, \varphi \rangle = \langle E, L^*\varphi \rangle \). Thus a distribution \( E \) is a solution of (5.95) if and only if

\[
\langle E, L^*\varphi \rangle = \varphi(\xi)
\] (5.96)

for each test function \( \varphi \).

Any two fundamental solutions for \( L \) with the same pole \( \xi \) differ by a solution of the homogeneous equation \( Lu = 0 \). Unless boundary conditions
are imposed, the homogeneous equation will have many solutions and the fundamental solution will not be uniquely determined. In most problems there will be grounds of symmetry or causality for selecting the particular fundamental solution which exhibits the appropriate physical behavior. Our usual approach will be to solve (5.95) on an intuitive basis using the familiar properties of the delta function. It then suffices to show that the function or distribution $E$ so obtained actually satisfies (5.96).

We also observe that if $L$ has constant coefficients, we can find the fundamental solution with pole at 0 [that is, $E(x \mid 0)$] and translate this solution to obtain the fundamental solution with pole at $x = \xi$. In this case,

$$E(x \mid \xi) = E(x - \xi \mid 0), \quad (5.97)$$

and the fundamental solution $E(x \mid 0)$ will be denoted simply by $E(x)$. Besides its intrinsic physical significance, the fundamental solution $E(x)$ enables us to solve the inhomogeneous equation

$$Lu = q, \quad (5.98)$$

where we shall assume that $q$ vanishes outside a finite sphere. We claim that one solution of this equation is

$$u = q * E, \quad (5.99)$$

where $*$ denotes the convolution product defined in (5.23). Let $u$ be given by (5.99); then, according to (5.26)

$$Lu = q * (LE) = q * \delta = q(x).$$

In many cases the fundamental solution $E$ is a function; we can then write (5.99) as an integral:

$$u(x) = \int_{\mathbb{R}^n} E(x - \xi)q(\xi)d\xi. \quad (5.100)$$

If $L$ does not have constant coefficients, we can no longer appeal to convolution products; instead one can often show that

$$u(x) = \int_{\mathbb{R}^n} E(x \mid \xi)q(\xi)d\xi \quad (5.101)$$

is a solution of (5.98).

**Laplace's Equation**

The "free-space" fundamental solution in three-dimensional potential theory satisfies

$$- \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \right) E = \delta(x),$$
or
\[-\nabla^2 E = \delta(x), \quad \text{or} \quad -\text{div grad } E = \delta(x). \quad (5.102)\]

Here \(E(x)\) can be interpreted as the electrostatic potential at an arbitrary observation point \(x\) due to a unit positive charge at \(x = 0\). Since the operator \(-\nabla^2\) is invariant under a rotation of coordinates, we shall look for a solution which depends only on \(r = |x|\). For \(r > 0\), \(E(r)\) satisfies the homogeneous equation \(\nabla^2 E = 0\), which takes the form
\[\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial E}{\partial r} \right) = 0.\]

Hence \(E = (C/r) + D\), and, if we require the potential to vanish at infinity, \(E = C/r\). To determine \(C\) we must take into account the magnitude of the source at \(x = 0\). Integrating (5.102) over a small sphere \(R_\varepsilon\) of radius \(\varepsilon\) and center at \(x = 0\), we obtain

\[-\int_{R_\varepsilon} \text{[div grad } E]\, dx = 1,
\]

and, using the divergence theorem,

\[-\int_{\sigma_\varepsilon} \left( \frac{\partial E}{\partial r} \right)_{r=\varepsilon} \, dS = 1,
\]

where \(\sigma_\varepsilon\) is the surface of \(R_\varepsilon\). Mathematically, this equation is the multidimensional equivalent of the jump condition on the derivative for one-dimensional problems [see (1.49), for instance]. Physically it expresses the conservation of charge: The flux of the electric field through the closed surface \(\sigma_\varepsilon\) is equal to the charge in the interior of \(\sigma_\varepsilon\).

Substituting \(E = C/r\), we find \(C = 1/4\pi\); hence

\[E(r) = \frac{1}{4\pi r}. \quad (5.103)\]

and, by (5.97),

\[E(x|\xi) = \frac{1}{4\pi|x - \xi|}. \quad (5.104)\]

The formulas (5.103) and (5.104) are, of course, the familiar ones for the electrostatic potential of a unit source in three dimensions.

For the two-dimensional version of (5.102), we have

\[\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial E}{\partial r} \right) = 0, \quad \text{for } r > 0,
\]

where \(r = |x| = (x_1^2 + x_2^2)^{1/2}\). A simple calculation shows that \(E(r) = C \log r + D\). We arbitrarily set \(D = 0\) and use the divergence theorem to find

\[E(r) = -\frac{1}{2\pi} \log r = \frac{1}{2\pi} \log \frac{1}{r}. \quad (5.105)\]
In applications to electrostatics, the electric field is observable but not the potential. Since the electric field involves only derivatives of the potential, the constant $D$ which we have summarily disposed of is in fact of no consequence.

The one-dimensional form of (5.102) is

$$-\frac{d^2 E}{dx^2} = \delta(x),$$

whose general solution is

$$E(x) = -\frac{1}{2}|x| + Ax + B.$$

If we require spherical symmetry, that is, $E = E(|x|)$, then $A = 0$. It is again convenient to set $B = 0$, so that

$$E(x) = -\frac{1}{2}|x|. \quad (5.106)$$

We now proceed to show that (5.103) is indeed a distributional solution of (5.102). We must show that, for every test function $\varphi(x)$, (5.96) is obeyed. Since $(\nabla^2)^* = \nabla^2$, we must prove that, for every $\varphi$,

$$\int_{R_3} \frac{1}{4\pi r} (\nabla^2 \varphi) dx = -\varphi(0).$$

The local integrability of $1/4\pi r$ implies that

$$\int_{R_3} \frac{1}{4\pi r} (\nabla^2 \varphi) dx = \lim_{\varepsilon \to 0} \int_{R_3 - R_\varepsilon} \frac{1}{4\pi r} (\nabla^2 \varphi) dx,$$

where $R_3 - R_\varepsilon$ is the region outside the sphere $R_\varepsilon$ previously introduced. By Green's theorem,

$$\int_{R_3 - R_\varepsilon} \frac{1}{4\pi r} (\nabla^2 \varphi) dx = \int_{R_3 - R_\varepsilon} \varphi \left( \nabla^2 \frac{1}{4\pi r} \right) dx$$

$$+ \int_{\partial R_\varepsilon} \left[ -\frac{1}{4\pi r} \frac{\partial \varphi}{\partial r} + \varphi \frac{\partial}{\partial r} \left( \frac{1}{4\pi r} \right) \right] dS,$$

where we have used the fact that $\varphi$ vanishes for sufficiently large $r$ in order to eliminate the surface integral at infinity. Since $\nabla^2 (1/4\pi r) = 0$ in $R_3 - R_\varepsilon$, we find

$$\int_{R_3 - R_\varepsilon} \frac{1}{4\pi r} (\nabla^2 \varphi) dx = -\int_{\partial R_\varepsilon} \left[ \frac{1}{4\pi r} \frac{\partial \varphi}{\partial r} + \frac{1}{4\pi r^2} \varphi \right] dS. \quad (5.107)$$

Since all derivatives of a test function are bounded in $R_3$, we have

$$\left| \frac{\partial \varphi}{\partial r} \right| < M \quad \text{for all } x,$$
hence

\[ \left| \int_{\sigma_e} \frac{1}{4\pi r} \frac{\partial \varphi}{\partial r} \, dS \right| \leq \frac{1}{4\pi e} (M)(4\pi e^2) = Me. \tag{5.108} \]

Also

\[
\int_{\sigma_e} \frac{\varphi(x)}{4\pi r^2} \, dS = \int_{\sigma_e} \frac{\varphi(0)}{4\pi r^2} \, dS + \int_{\sigma_e} \frac{\varphi(x) - \varphi(0)}{4\pi r^2} \, dS,
\]

\[
= \varphi(0) + \int_{\sigma_e} \frac{\varphi(x) - \varphi(0)}{4\pi r^2} \, dS,
\]

\[
\left| \int_{\sigma_e} \frac{\varphi(x) - \varphi(0)}{4\pi r^2} \, dS \right| \leq \max_{x \in \sigma_e} |\varphi(x) - \varphi(0)|.
\]

Since \(\varphi(x)\) is continuous at \(x = 0\),

\[
\lim_{\varepsilon \to 0} \left[ \max_{x \in \sigma_e} |\varphi(x) - \varphi(0)| \right] = 0,
\]

and, using (5.108), we find from (5.107) that

\[
\lim_{\varepsilon \to 0} \int_{R_3 - R_\varepsilon} \frac{1}{4\pi r} (\nabla^2 \varphi) \, dx = -\varphi(0),
\]

which completes the proof that (5.103) is a distributional solution of (5.102).

The fundamental solution \(E_n(x)\) for the \(n\)-dimensional Laplace operator \(-\nabla^2\) with pole at \(x = 0\) satisfies

\[
-\nabla^2 E_n = -\left( \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right) E_n = \delta(x). \tag{5.109}
\]

If we look for a solution which depends only on \(r = (x_1^2 + \cdots + x_n^2)^{1/2}\), (5.109) can be reduced to an ordinary differential equation in \(r\). Indeed, if a function \(u\) depends only on the radial coordinate \(r\),

\[
\frac{\partial u}{\partial x_i} = \left( \frac{\partial u}{\partial r} \right) \frac{\partial r}{\partial x_i} = \frac{x_i}{r} \left( \frac{\partial u}{\partial r} \right),
\]

\[
\frac{\partial^2 u}{\partial x_i^2} = \frac{x_i^2}{r^2} \frac{\partial^2 u}{\partial r^2} - \frac{x_i^2}{r^3} \frac{\partial u}{\partial r} + \frac{1}{r} \frac{\partial u}{\partial r},
\]

\[
\nabla^2 u = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2} = \frac{\partial^2 u}{\partial r^2} - \frac{1}{r} \frac{\partial u}{\partial r} + \frac{n}{r^2} \frac{\partial u}{\partial r} = \frac{\partial^2 u}{\partial r^2} + \frac{n - 1}{r} \frac{\partial u}{\partial r}.
\]

Thus for \(x \neq 0\), (5.109) becomes the homogeneous equation

\[
\frac{\partial^2 E_n}{\partial r^2} + \frac{n - 1}{r} \frac{\partial E_n}{\partial r} = \frac{1}{r^{n-1}} \frac{\partial}{\partial r} \left( r^{n-1} \frac{\partial E_n}{\partial r} \right) = 0. \tag{5.110}
\]
Therefore,

\[ E_n = \frac{C}{\rho^{n-2}} + D, \quad n > 2 \]

and, if we require the potential to vanish at large distances from the source, we must set \( D = 0 \). Integrating (5.109) over a small sphere \( R_\varepsilon \) of radius \( \varepsilon \) and center at \( x = 0 \), we obtain

\[ \int_{R_\varepsilon} [\text{div grad } E_n] dx = -1. \]

The divergence theorem then gives

\[ \int_{\sigma_\varepsilon} \left( \frac{\partial E_n}{\partial r} \right) dS = -1, \]

where \( \sigma_\varepsilon \) is the surface of the sphere \( R_\varepsilon \).

Substituting \( E_n = C/r^{n-2} \), we find \( C = 1/(n - 2)S_n(1) \), where \( S_n(1) \) is the surface area of the \( n \)-dimensional sphere of unit radius. By (5.18) we have

\[ E_n(r) = \frac{1}{(n - 2)S_n(1)r^{n-2}} = \frac{[(n/2) - 1]!}{2(\pi^{n/2})(n - 2)r^{n-2}}, \quad n > 2. \tag{5.111} \]

It is a simple exercise to verify that (5.111) is, in fact, a distributional solution of (5.109).

**Helmholtz Equation**

We want the fundamental solution for the Helmholtz operator \(-\nabla^2 - \lambda\), where \( \lambda \) is an arbitrary complex number. The fundamental solution \( E_n(x; \lambda) \) satisfies

\[ -\nabla^2 E_n - \lambda E_n = \delta(x) \tag{5.112} \]

or

\[ -\nabla^2 E_n + \mu^2 E_n = \delta(x), \tag{5.113} \]

where

\[ \mu^2 = -\lambda. \]

As usual we define \( \sqrt{\lambda} \) to be the unambiguous square root of \( \lambda \) which has nonnegative imaginary part, that is \( \sqrt{\lambda} = \alpha + i\beta \), where \( \beta \geq 0 \) and \( \beta = 0 \) if and only if \( \lambda \) is in \([0, \infty)\).† It will then be convenient to let

\[ \mu = -i\sqrt{\lambda}, \]

so that \( \mu^2 = -\lambda \) and \( \mu \) is real positive when \( \lambda \) is real negative.

† The symbol \([0, \infty)\) stands for the set of real nonnegative numbers.
Again we shall require that $E_n$ be spherically symmetric; then, for $x \neq 0$, $E_n$ must satisfy

$$
\frac{d}{dr} \left( r^{n-1} \frac{du}{dr} \right) + \lambda r^{n-1} u = 0. \tag{5.114}
$$

Equation (5.114) can be reduced to one of the Bessel type by making the substitution $u = wr^{1-(n/2)}$. A straightforward calculation shows that $w$ satisfies

$$
\frac{d}{dr} \left( r \frac{dw}{dr} \right) - \frac{w}{r} \left( 1 - \frac{n}{2} \right)^2 + \lambda rw = 0.
$$

This is Bessel's equation of order $(n/2) - 1$ with parameter $\lambda$, whose general solution is conveniently written in terms of Hankel functions:

$$
w(r) = C_1 H_{(n/2)-1}^{(1)}(\sqrt{\lambda}r) + C_2 H_{(n/2)-1}^{(2)}(\sqrt{\lambda}r), \quad n \geq 2.
$$

The corresponding general solution of (5.114) is

$$
u(r) = r^{1-(n/2)} [C_1 H_{(n/2)-1}^{(1)}(\sqrt{\lambda}r) + C_2 H_{(n/2)-1}^{(2)}(\sqrt{\lambda}r)]. \tag{5.115}
$$

If $\lambda$ is not in $[0, \infty)$, $\sqrt{\lambda}$ has positive imaginary part and the solution $H_{(n/2)-1}^{(2)}(\sqrt{\lambda}r)$ is exponentially large at $r = \infty$, whereas $H_{(n/2)-1}^{(1)}(\sqrt{\lambda}r)$ is exponentially small. Since we want $E_n$ to vanish at $r = \infty$, we have

$$
E_n = Cr^{1-(n/2)}H_{(n/2)-1}^{(1)}(\sqrt{\lambda}r). \tag{5.116}
$$

Applying the divergence theorem to (5.112), we obtain

$$
\int_{\sigma_n} \frac{\partial E_n}{\partial r} dS = -1
$$
or

$$
\lim_{r \to 0} r^{n-1} S_n(1) \frac{\partial E_n}{\partial r} = -1. \tag{5.117}
$$

For small values of $r$, we have the asymptotic form

$$H_n^{(1)}(r) \sim -\frac{i2^n(n-1)!}{\pi} r^{-n},
$$

so that insertion of (5.116) into (5.117) yields

$$C = \frac{i\pi 2^{-n/2}(\sqrt{\lambda})^{(n-2)/2}}{[(n/2)-1]!S_n(1)} = \frac{i}{4} \left( \frac{\sqrt{\lambda}}{2\pi} \right)^{(n-2)/2}.
$$

Thus for $n \geq 2$ and $\lambda$ not in $[0, \infty)$, the required solution of (5.112) is

$$E_n(r; \lambda) = \frac{i}{4} \left( \frac{\sqrt{\lambda}}{2\pi r} \right)^{(n-2)/2} H_{(n/2)-1}^{(1)}(\sqrt{\lambda}r), \quad n \geq 2. \tag{5.118}
$$
It is sometimes more convenient to let $\lambda = -\mu^2$, $\mu = -i\sqrt{\lambda}$. Since $\mu$ has positive real part, we can use the formula (see Appendix B, Volume I)

$$H^{(1)}_{(n/2)-1}(i\mu r) = \frac{2}{\pi i} \left(-i\right)^{(n-2)/2} K_{(n/2)-1}(\mu r),$$

to obtain

$$E_n(r; -\mu^2) = \frac{1}{2\pi} \left(\frac{\mu}{2\pi r}\right)^{(n-2)/2} K_{(n/2)-1}(\mu r), \quad (5.119)$$

which holds whenever $-\mu^2$ is not in $[0, \infty)$, that is, for all $\mu$ with Re $\mu > 0$.

For the case $n = 2$, (5.118) and (5.119) become

$$E_2(r; \lambda) = \frac{i}{4} H_0^{(1)}(\sqrt{\lambda} r) = \frac{1}{2\pi} K_0(\mu r). \quad (5.120)$$

Using

$$H_{1/2}^{(1)}(z) = \frac{1}{i} (2/\pi)^{1/2} \frac{e^{iz}}{z^{1/2}},$$

we have for the case $n = 3$,

$$E_3(r; \lambda) = \frac{e^{i\sqrt{\lambda} r}}{4\pi r} = \frac{e^{-\mu r}}{4\pi r}. \quad (5.121)$$

A direct calculation shows that the one-dimensional fundamental solution is

$$E_1(x; \lambda) = \frac{i e^{i\sqrt{\lambda} |x|}}{2\sqrt{\lambda}} = \frac{e^{-\mu |x|}}{2\mu}. \quad (5.122)$$

Just as for Laplace’s equation, one can again verify that (5.118) is a distributional solution of (5.112). The solution (5.118) was derived for $\lambda$ not in $[0, \infty)$. Since the case $\lambda = 0$ was previously treated, it remains to investigate the problem for real positive values of $\lambda$. In this case both solutions in (5.115) vanish at $r = \infty$ like $r^{1/2}r^{-n/2}$ and there is no natural mathematical criterion for distinguishing between them. To obtain guidance in making an intelligent choice between these solutions we must examine the specific physical problem from which (5.112) arises. This is done at a later stage in the study of the wave equation.

We now proceed to obtain an alternative representation for the solution (5.118). Let us first consider the case $\lambda = -1$, so that (5.112) becomes

$$-\nabla^2 E_n(x; -1) + E_n(x; -1) = -\sum_{k=1}^{n} \frac{\partial^2 E_n}{\partial x_k^2} + E_n = \delta(x_1)\delta(x_2) \cdots \delta(x_n). \quad (5.123)$$
The $n$-dimensional Fourier transform of $E_n(x; -1)$ is defined by

$$E_n^\wedge(\alpha; -1) = \int_{R^n} e^{i\alpha \cdot x} E_n(x; -1) dx,$$

(5.124)

with the inversion formula

$$E_n(x; -1) = \frac{1}{(2\pi)^n} \int_{R^n} e^{-i\alpha \cdot x} E_n^\wedge(\alpha; -1) d\alpha.$$

(5.125)

We easily find $E_n^\wedge$ by multiplying (5.123) by $e^{i\alpha \cdot x}$ and integrating over all of $n$-dimensional space. Taking into account the fact that $E_n$ vanishes exponentially at $\infty$, we obtain

$$(\alpha_1^2 + \cdots + \alpha_n^2 + 1)E_n^\wedge = 1,$$

so that

$$E_n^\wedge = \frac{1}{\|\alpha\|^2 + 1},$$

(5.126)

where $\|\alpha\|^2 = \alpha_1^2 + \cdots + \alpha_n^2$. From (5.125), we have

$$E_n(x; -1) = \frac{1}{(2\pi)^n} \int_{R^n} \frac{e^{-i\alpha \cdot x}}{\|\alpha\|^2 + 1} d\alpha.$$

(5.127)

Unfortunately this integral diverges because the integrand does not tend to zero sufficiently fast at infinity to compensate for the factor $\|\alpha\|^{n-1}$ which appears in the volume element $d\alpha$. We shall use purely formal manipulations to calculate the integral (5.127) and leave to the reader the verification that the result so obtained satisfies all the conditions on a fundamental solution.

Let us introduce the Cartesian coordinates $\alpha_1, \ldots, \alpha_n$, so that $\alpha_n$ is in the direction of the fixed vector $x$. Then on the hyperplane $\alpha_n = \text{constant}$ we have $e^{-i\alpha \cdot x} = e^{-i\alpha_n|x|}$. On this hyperplane consider the annular region $\Delta$ between the concentric spheres $\alpha_1^2 + \cdots + \alpha_n^2 = \rho^2$ and $\alpha_1^2 + \cdots + \alpha_n^2 = (\rho + \Delta \rho)^2$. The volume of $\Delta$ is $S_{n-1}(\rho)\rho \Delta \rho$ or $\rho^{n-2}S_{n-1}(1)\rho \Delta \rho$, where $S_{n-1}(1)$ is the surface area of a unit sphere in $n-1$ dimensions. The integrand in (5.127) is constant over $\Delta$ and has the value $e^{-i\alpha_n|x|}/\rho^2 + \alpha_n^2 + 1$. Summing the contributions from such annular regions and then integrating over $\alpha_n$, we obtain

$$E_n(x; -1) = \frac{1}{(2\pi)^n} \int_{0}^{\infty} d\rho \int_{-\infty}^{\infty} d\alpha_n \frac{e^{-i\alpha_n|x|}}{\rho^2 + \alpha_n^2 + 1} \rho^{n-2}S_{n-1}(1).$$

The integration over $\alpha_n$ is easily performed. The integrand is exponentially small as $|\alpha_n| \to \infty$ in the lower half of the complex $\alpha_n$ plane. The only singularity in the lower half-plane is a simple pole at $\alpha_n = -i(\rho^2 + 1)^{1/2}$. By Cauchy's integral theorem,

$$\int_{-\infty}^{\infty} \frac{e^{-i\alpha_n|x|}}{\rho^2 + \alpha_n^2 + 1} d\alpha_n = \frac{\pi}{(\rho^2 + 1)^{1/2}} e^{-|x|(\rho^2+1)^{1/2}}.$$


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Thus
\[ E_n(x; -1) = \frac{\pi S_{n-1}(1)}{(2\pi)^n} \int_0^\infty \frac{\rho^{n-2}}{(\rho^2 + 1)^{1/2}} e^{-|x|(\rho^2 + 1)^{1/2}} d\rho, \]
and setting \( \rho^2 + 1 = z^2, 1 < z < \infty, \)
\[ E_n(x; -1) = \frac{\pi S_{n-1}(1)}{(2\pi)^n} \int_1^\infty e^{-|x|z}(z^2 - 1)^{(n-3)/2} dz. \]  
(5.128)

Now let \( \lambda = -\mu^2, \) where \( \mu^2 \) is a real positive number and \( \mu \) is its real positive square root. The fundamental solution \( E_n(x; -\mu^2) \) satisfies
\[-\nabla^2 E_n + \mu^2 E_n = \delta(x).\]

The change of variables \( x'_i = \mu x_i \) transforms this equation to
\[-\mu^2 \nabla^2 E_n + \mu^2 E_n = \delta\left(\frac{x'}{\mu}\right) = \mu^n \delta(x'),\]
where the \( \nabla^2 \) operation is now with respect to \( x' \). We conclude that
\[ E_n(x; -\mu^2) = \mu^{n-2} E_n(\mu x; -1), \]
and, after setting \( |x| = r, \)
\[ E_n(r; -\mu^2) = \frac{\pi \mu^{n-2} S_{n-1}(1)}{(2\pi)^n} \int_1^\infty e^{-\mu r z}(z^2 - 1)^{(n-3)/2} dz. \]  
(5.129)

Both the expressions (5.129) and (5.119) hold for \( \mu \) real positive and both are analytic functions of the complex variable \( \mu \) for \( \text{Re} \mu > 0. \) Since (5.119) holds for all \( \mu \) such that \( \text{Re} \mu > 0, \) so does (5.129).

If we differentiate (5.129) with respect to \( r \) and then integrate by parts, we find the recursion formula
\[ -\frac{1}{2\pi r} \frac{\partial E_n}{\partial r} = E_{n+2}. \]  
(5.130)

Moreover, for the particular cases \( n = 2 \) and \( n = 3, \) we observe that (5.129) yields
\[ E_2(r; -\mu^2) = \frac{1}{2\pi} \int_1^\infty e^{-\mu r z}(z^2 - 1)^{-1/2} dz, \]
\[ E_3(r; -\mu^2) = \frac{e^{-\mu r}}{4\pi r}, \]
so that, using (5.130),

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\[ E_n(r; -\mu^2) = \begin{cases} \left( -\frac{1}{2\pi r} \frac{\partial}{\partial r} \right)^{(n-1)/2} \left[ \frac{e^{-\mu r}}{2\mu} \right], & n \text{ odd;} \\ \left( -\frac{1}{2\pi r} \frac{\partial}{\partial r} \right)^{(n-2)/2} \left[ \frac{1}{2\mu} \int_1^\infty e^{-\mu r}(z^2 - 1)^{-1/2} \, dz \right], & n \text{ even.} \end{cases} \] (5.131)

We observe that the formula is also correct for \( n = 1 \).

**Causal Fundamental Solution for the Heat Equation**

We now consider partial differential equations which describe time-dependent physical phenomena. Suppose there are \( n \) space coordinates \( x_1, \ldots, x_n \) and the time coordinate \( t \); the symbol \( x \) is used as an abbreviation for the position vector in space and \( (x, t) \) describes a point in space-time. We could of course have included \( t \) as an undistinguished coordinate in a space of \( n + 1 \) dimensions; it is preferable to single out the time coordinate \( t \) because the behavior in time will be quite different from the behavior in the other coordinates.

Let us first investigate the \( n \)-dimensional equation of heat conduction,

\[ \frac{\partial u}{\partial t} - a\nabla^2 u = q(x, t), \quad (5.132) \]

where the operator \( \nabla^2 \) is understood to act *only* on the space coordinates; that is,

\[ \nabla^2 u = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2}. \]

In (5.132), \( u(x, t) \) is the temperature in a homogeneous isotropic medium which we take to be of infinite extent (filling the entire \( n \)-dimensional physical space), \( a \) is a positive constant, and \( q(x, t) \) is the density of the external heat sources; that is, \( q(x, t) \) is the heat input into the medium per unit volume per unit time.

A fundamental solution \( E(x, t | \xi, \tau) \) satisfies

\[ \frac{\partial E}{\partial t} - a\nabla^2 E = \delta(x - \xi)\delta(t - \tau). \quad (5.133) \]

Since the coefficients in the differential operator are constants, it will suffice to consider the solution \( E(x, t) = E(x, t | 0, 0) \), which satisfies

\[ \frac{\partial E}{\partial t} - a\nabla^2 E = \delta(x)\delta(t), \quad (5.134) \]

and then use the relation

\[ E(x, t | \xi, \tau) = E(x - \xi, t - \tau). \]
DEFINITION. The causal fundamental solution $C(x, t)$ is the particular solution of (5.134) which vanishes identically for $t < 0$. Thus $C(x, t)$ satisfies
\[
\frac{\partial C}{\partial t} - a\nabla^2 C = \delta(x)\delta(t), \quad C \equiv 0 \text{ for } t < 0. \tag{5.135}
\]

The causal fundamental solution $C(x, t)$ has a direct physical interpretation; it is the temperature distribution in a medium which is at zero temperature up to time $t = 0$, when a concentrated source is introduced at $x = 0$, this source instantaneously releasing a unit of heat. Although $C$ is defined for all $t$ and $x$, its calculation presents a problem only for $t > 0$ ($C = 0$ for $t < 0$). This immediately suggests a slightly different point of view; for $t > 0$ no sources are present, so that $C$ satisfies the homogeneous equation and must reduce, at $t = 0+$, to a certain initial temperature. This initial temperature is the one to which the medium has been raised just after the introduction of an instantaneous concentrated source of unit strength. We now show that this initial temperature is $\delta(x)$.

Theorem. The causal fundamental solution coincides for $t > 0$ with the solution of the initial value problem
\[
\frac{\partial u}{\partial t} - a\nabla^2 u = 0, \quad t > 0; \quad u(x, 0+) = \delta(x). \tag{5.136}
\]

Proof. Let $u(x, t)$ be the solution for $t > 0$ of system (5.136) and let $C(x, t)$ be defined by
\[
C(x, t) = \begin{cases} 
  u(x, t), & t > 0; \\
  0, & t < 0. 
\end{cases} \tag{5.137}
\]
We must show that $C(x, t)$ satisfies (5.135). The requirement $C \equiv 0$ for $t < 0$ is obviously satisfied. We can write
\[
C(x, t) = H(t)u(x, t),
\]
where $H(t)$ is the Heaviside function which is 1 for $t > 0$ and vanishes for $t < 0$. Then, proceeding formally,
\[
\nabla^2 C = H(t)\nabla^2 u,
\]
\[
\frac{\partial C}{\partial t} = H(t)\frac{\partial u}{\partial t} + u(x, t) \frac{dH}{dt}(t) = H(t)\frac{\partial u}{\partial t} + u(x, 0+)\delta(t).
\]
Thus
\[
\frac{\partial C}{\partial t} - a\nabla^2 C = u(x, 0+)\delta(t) = \delta(x)\delta(t),
\]
which completes the proof.
We now calculate the causal solution $C_n(x, t)$ for the case of $n$ space dimensions. Taking the Fourier transform of (5.135) over the space coordinates, we obtain

$$\frac{dC_n^\wedge}{dt} + a\|\alpha\|^2 C_n^\wedge = \delta(t); \quad C_n^\wedge \equiv 0 \text{ for } t < 0,$$

(5.138)

where

$$C_n^\wedge = \int_{R^n} e^{i\alpha \cdot x} C_n(x, t) dx,$$

$$\|\alpha\|^2 = \alpha_1^2 + \cdots + \alpha_n^2.$$ 

From (5.138), we find for $t > 0$,

$$C_n^\wedge = e^{-a\|\alpha\|^2 t}.$$

(5.139)

The inversion formula

$$C_n = \frac{1}{(2\pi)^n} \int_{R^n} e^{-i\alpha \cdot x} C_n^\wedge(\alpha, t) d\alpha$$

yields

$$C_n = \prod_{k=1}^n \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-is_k\alpha_k} e^{-a\alpha_k^2 t} d\alpha_k \right].$$

Using the well-known formula

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\beta x} e^{-a\beta^2 t} d\beta = \frac{1}{(4\pi at)^{1/2}} e^{-x^2 / 4at},$$

we obtain

$$C_n(x, t) = \prod_{k=1}^n (4\pi at)^{-1/2} e^{-x_k^2 / 4at}.$$ 

With $x_1^2 + \cdots + x_n^2 = r^2$, this becomes

$$C_n(r, t) = (4\pi at)^{-n/2} e^{-r^2 / 4at}, \quad t > 0.$$

(5.140)

Of course for $t < 0$, $C_n \equiv 0$. As could have been surmised from the beginning, $C$ has no angular space dependence. The result (5.140) could have been obtained by a Laplace transform over the time (see Exercise 5.36). It is easy to verify that (5.140) satisfies (5.136). For $t > 0$, $C_n(r, t)$ has continuous derivatives and is a strict solution of the differential equation, as can be seen by substitution. As $t \to 0^+$, $C_n(r, t)$ approaches $\delta(x)$ in the distributional sense (see Exercise 5.3).
Nondissipative or Undamped Wave Equation

We now turn our attention to the wave equation without dissipation,

$$\frac{\partial^2 u}{\partial t^2} - c^2 \nabla^2 u = q(x, t).$$  \hspace{1cm} \text{(5.141)}

The causal fundamental solution $C(x, t)$ satisfies

$$\frac{\partial^2 C}{\partial t^2} - c^2 \nabla^2 C = \delta(x)\delta(t), \quad C \equiv 0 \text{ for } t < 0.$$  \hspace{1cm} \text{(5.142)}

Again we can show that $C$ is equally well characterized, for $t > 0$, by an initial value problem for the homogeneous wave equation. Since the equation is now of second order in the time coordinate, it is not surprising that initial values will have to be prescribed for both $C$ and $\partial C/\partial t$. We now show that $C(x, 0+) = 0$ and $\partial C(x, 0+)/\partial t = \delta(x)$.

Let $u(x, t)$ be the solution of the initial value problem

$$\frac{\partial^2 u}{\partial t^2} - c^2 \nabla^2 u = 0, \quad t > 0; \quad u(x, 0+) = 0; \quad \frac{\partial u}{\partial t} (x, 0+) = \delta(x),$$

and let $C(x, t)$ be defined as $u(x, t)H(t)$, where $H(t)$ is the Heaviside function. We want to prove that $C$ satisfies (5.142). It is clear that $C \equiv 0$ for $t < 0$, and

$$\nabla^2 C = \nabla^2 uH = H(t)\nabla^2 u,$$

$$\frac{\partial C}{\partial t} = H(t) \frac{\partial u}{\partial t} + u(x, 0+)\delta(t),$$  \hspace{1cm} \text{(5.143)}

$$\frac{\partial^2 C}{\partial t^2} = H(t) \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} (x, 0+)\delta(t) + u(x, 0+)\delta'(t).$$  \hspace{1cm} \text{(5.144)}

Since $u(x, 0+) = 0$ and $\partial u(x, 0+)/\partial t = \delta(x)$, we have

$$\frac{\partial^2 C}{\partial t^2} - c^2 \nabla^2 C = \delta(x)\delta(t),$$

which is the desired result.†

We proceed with the calculation of the causal fundamental solution $C_n(x, t)$ for the wave equation in $n$ space dimensions. We first set $c^2 = 1$ and later treat the case of arbitrary $c^2$. To illustrate a different method of

† Equation (5.143) can also be written $\partial C/\partial t = H(t)(\partial u/\partial t) + u(x, t)\delta(t)$, so that

$$\frac{\partial^2 C}{\partial t^2} = H(t) \frac{\partial^2 u}{\partial t^2} + 2 \frac{\partial u}{\partial t} (x, t)\delta(t) + u(x, t)\delta'(t).$$

We can now appeal to the equation $f(t)\delta'(t) = f(0)\delta'(t) - f'(0)\delta(t)$ to show that $\partial^2 C/\partial t^2$ is still given by (5.144).
approach, we apply a Laplace transform on the time coordinate. The initial value problem satisfied by \( C_n(x, t) \) is
\[
\frac{\partial^2 C_n}{\partial t^2} - \nabla^2 C_n = 0, \quad t > 0; \quad C_n(x, 0+) = 0; \quad \frac{\partial C_n}{\partial t} (x, 0+) = \delta(x).
\]
(5.145)

We multiply the equation by \( e^{-st} \) and integrate from 0 to \( \infty \) to obtain
\[
\int_0^\infty e^{-st} \frac{\partial^2 C_n}{\partial t^2} \, dt - \nabla^2 \int_0^\infty e^{-st} C_n \, dt = 0.
\]
After integration by parts and use of the initial conditions, this equation becomes
\[
s^2 \mathcal{C}_n - \nabla^2 \mathcal{C}_n = \delta(x),
\]
(5.146)
where
\[
\mathcal{C}_n = \int_0^\infty e^{-st} C_n(x, t) dt.
\]
The term \( \delta(x) \) in (5.146) arises from the initial condition on \( \partial C_n/\partial t \) at \( t = 0^+ \). We could just as well have started from (5.142) with \( c^2 = 1 \) and then integrated from 0 to \( \infty \); now \( C_n(x, 0-) \) and \( \partial C_n(x, 0-) / \partial t \) vanish but there is a contribution from the inhomogeneous term in (5.142) and again (5.146) is obtained.

According to (5.131), we have, for \( \text{Re} \ s > 0 \),
\[
\mathcal{C}_n = \begin{cases} 
\left( - \frac{1}{2\pi r} \frac{\partial}{\partial r} \right)^{n-1/2} \frac{e^{-sr}}{2s}, & n \text{ odd}; \\
\left( - \frac{1}{2\pi r} \frac{\partial}{\partial r} \right)^{(n-2)/2} \left[ \int_1^\infty e^{-srz} (z^2 - 1)^{-1/2} \frac{dz}{2\pi} \right]; & n \text{ even}.
\end{cases}
\]
(5.147)

A simple application of the inversion formula for the Laplace transform shows that \( e^{-sr}/2s \) is the Laplace transform of \( \frac{1}{2} H(t - r) \) and that \( e^{-srz} \) is the Laplace transform of \( \delta(t - rz) \). Since the differentiations and integrations in (5.147) are with respect to variables other than \( s \), we can perform these operations after the inversion. Hence, for \( n \text{ odd} \),
\[
C_n(r, t) = \left( - \frac{1}{2\pi r} \frac{\partial}{\partial r} \right)^{(n-1)/2} \left[ \frac{1}{2} H(t - r) \right]
= \left( \frac{1}{2\pi r} \frac{\partial}{\partial t} \right)^{(n-1)/2} \left[ \frac{1}{2} H(t - r) \right]
= \left( \frac{1}{2\pi r} \frac{\partial}{\partial t} \right)^{(n-3)/2} \left[ \frac{1}{4\pi r} \delta(t - r) \right].
\]
(5.148)
For \( n \) even,
\[
C_n(r, t) = \left( -\frac{1}{2\pi r} \frac{\partial}{\partial r} \right)^{(n-2)/2} \left[ \frac{1}{2\pi} \int_1^\infty \delta(t - rz)(z^2 - 1)^{-1/2} \, dz \right]
= H(t - r) \left( -\frac{1}{2\pi r} \frac{\partial}{\partial r} \right)^{(n-2)/2} \left[ \frac{1}{2\pi r} \left( \frac{t^2}{r^2} - 1 \right) \right]^{-1/2}
\]  
(5.149)

The particular cases \( n = 1, 2, 3 \), are of special importance. We find
\[
C_1(r, t) = \frac{1}{2} H(t - r)
\]  
(5.150)
\[
C_2(r, t) = \frac{1}{2\pi} H(t - r)(t^2 - r^2)^{-1/2}
\]  
(5.151)
\[
C_3(r, t) = \frac{1}{4\pi r} \delta(t - r).
\]  
(5.152)

In all cases \( C_n \) is spherically symmetric in the space variables and vanishes for \( t < r \). Thus the front of the disturbance propagates with unit velocity. An interesting distinction is apparent between odd- and even-dimensional problems. In the odd cases (for \( n \geq 3 \)), the entire disturbance is concentrated on the sphere \( r = t \), whereas for even-dimensional problems, a wake is present after the front reaches the observation point. To verify our results, we may either show that \( C_n \) satisfies (5.142) or (5.145). The latter is easier to verify, but we must first assign a rigorous distributional meaning to (5.145). We shall regard \( t \) as a parameter and \( C_n(x, t) \) as a distribution in the space coordinates depending parametrically on \( t \). The initial value problem (5.145) is then understood to mean

\[
\frac{d^2}{dt^2} \langle C_n(x, t), \varphi(x) \rangle = \langle C_n(x, t), \nabla^2 \varphi(x) \rangle, \quad \text{for } t > 0;
\]

\[
\lim_{t \to 0^+} \langle C_n(x, t), \varphi(x) \rangle = 0;
\]  
(5.153)
\[
\lim_{t \to 0^+} \frac{d}{dt} \langle C_n(x, t), \varphi(x) \rangle = \varphi(0).
\]

Consider first the case \( n = 1 \); we want to show that \( C_1 = \frac{1}{2} H(t - |x|), \) \( t > 0 \), satisfies (5.153). We have
\[
\langle C_1, \varphi(x) \rangle = \frac{1}{2} \int_{-t}^t \varphi(x) \, dx,
\]
\[
\frac{d}{dt} \langle C_1, \varphi \rangle = \frac{1}{2} \varphi(t) + \frac{1}{2} \varphi(-t),
\]
\[
\frac{d^2}{dt^2} \langle C_1, \varphi \rangle = \frac{1}{2} \varphi'(t) - \frac{1}{2} \varphi'(-t).
\]
Since
\[ \left\langle C_1, \frac{d^2 \varphi}{dx^2} \right\rangle = \frac{1}{2} \int_{-t}^{t} \frac{d^2 \varphi}{dx^2} \, dx = \frac{1}{2} \varphi'(t) - \frac{1}{2} \varphi'(-t), \]
the first equation in (5.153) is satisfied. Moreover,
\[
\lim_{t \to 0^+} \langle C_1, \varphi \rangle = \lim_{t \to 0^+} \frac{1}{2} \int_{-t}^{t} \varphi(x) \, dx = 0,
\]
\[
\lim_{t \to 0^+} \frac{d}{dt} \langle C_1, \varphi \rangle = \lim_{t \to 0^+} \left[ \frac{1}{2} \varphi(t) + \frac{1}{2} \varphi(-t) \right] = \varphi(0),
\]
which completes the verification.

Turning to \( C_3(x, t) \), we first recall the definition of (5.152):
\[ \langle C_3, \varphi(x) \rangle = \frac{1}{4\pi t} \int_{\sigma_t} \varphi(x) \, dS, \]
where \( \sigma_t \) is the surface of the three-dimensional sphere of radius \( t \) with center at \( x = 0 \). Thus
\[ \langle C_3, \varphi(x) \rangle = t \varphi_{av}(t), \quad (5.154) \]
where \( \varphi_{av}(t) \) is the average of \( \varphi(x) \) on \( \sigma_t \).

Clearly
\[
\lim_{t \to 0^+} \langle C_3, \varphi \rangle = 0
\]
\[
\lim_{t \to 0^+} \left\langle \frac{dC_3}{dt}, \varphi \right\rangle = \varphi(0),
\]
so that the last two requirements of (5.153) are satisfied. Further,
\[ \langle C_3, \nabla^2 \varphi \rangle = \frac{1}{4\pi t} \int_{\sigma_t} \nabla^2 \varphi(x) \, dS, \]
which, by using spherical coordinates, reduces to
\[ \langle C_3, \nabla^2 \varphi \rangle = \frac{1}{t^2} \frac{d}{dt} \left[ t^2 \frac{d}{dt} \varphi_{av}(t) \right] = 2 \varphi_{av}'(t) + t \varphi_{av}''(t). \]

On the other hand, from (5.154),
\[ \frac{d^2}{dt^2} \langle C_3, \varphi \rangle = 2 \varphi_{av}'(t) + t \varphi_{av}''(t), \]
which completes the proof.

It remains to consider the case where \( c^2 \neq 1 \). The substitution \( t' = ct \) reduces (5.142) to
\[ c^2 \frac{\partial^2 C}{\partial t'^2} - c^2 \nabla^2 C = \delta(x) \delta \left( \frac{t'}{c} \right) = c \delta(x) \delta(t'), \]
so that, for \( n \) odd, the solution of (5.142) is
\[
C_n(r, t) = \frac{1}{c} \left( -\frac{1}{2\pi r} \frac{\partial}{\partial r} \right)^{(n-1)/2} \left[ \frac{1}{2}\mathcal{H}(ct - r) \right] 
\]
(5.155)

whereas for \( n \) even,
\[
C_n(r, t) = \frac{H(ct - r)}{c} \left( -\frac{1}{2\pi r} \frac{\partial}{\partial r} \right)^{(n-2)/2} \left[ \frac{1}{2\pi r} \left( \frac{c^2 t^2}{r^2} - 1 \right)^{-1/2} \right] 
\]
(5.156)

It is clear that the front of the disturbance now propagates with velocity \( c \) and we are therefore entitled to call \( c \) the velocity of propagation.

**Wave Equation with Dissipation**

If \( c^2 = 1 \) and the medium is dissipative, the inhomogeneous wave equation has the form
\[
\frac{\partial^2 u}{\partial t^2} + 2\gamma \frac{\partial u}{\partial t} - \nabla^2 u = q(x, t),
\]
(5.157)

where \( \gamma \) is a positive constant. We also refer to (5.157) as the *damped wave equation*.

The separable solutions of the homogeneous equation have the time dependence \( e^{-\gamma t} \exp[\pm t(\gamma^2 - \alpha^2)^{1/2}] \), where \( \alpha \) is the arbitrary separation parameter. Since the factor \( e^{-\gamma t} \) is common to all such solutions, we shall make the preliminary transformation
\[
u(x, t) = w(x, t)e^{-\gamma t},
\]
which reduces (5.157) to the form
\[
\frac{\partial^2 w}{\partial t^2} - \gamma^2 w - \nabla^2 w = q'(x, t),
\]
(5.158)

where
\[q'(x, t) = e^{\gamma t} q(x, t).\]

The equation (5.158) is known as the *equation of telegraphy*. Its causal fundamental solution satisfies
\[
\frac{\partial^2 C}{\partial t^2} - \gamma^2 C - \nabla^2 C = \delta(x)\delta(t), \quad C \equiv 0 \text{ for } t < 0. 
\]
(5.159)

Taking the Laplace transform with respect to the time coordinate, we obtain
\[
(s^2 - \gamma^2)\tilde{C} - \nabla^2 \tilde{C} = \delta(x). 
\]
(5.160)

Thus \( \tilde{C} \) is the fundamental solution for the Helmholtz equation with parameter \( s^2 - \gamma^2 \).
We first examine the case \( n = 1, \gamma^2 = 1 \). Then

\[
\tilde{C}_1 = \frac{e^{-(s^2-1)^{1/2}|x|}}{2(s^2 - 1)^{1/2}},
\]

(5.161)

where \((s^2 - 1)^{1/2}\) is to be an analytic function of \( s \) for \( \text{Re} \ s > 1 \) and is real positive for those real \( s \) which are greater than 1. These last two requirements guarantee that \( \tilde{C}_1 \) is indeed a Laplace transform! We now proceed to define \((s^2 - 1)^{1/2}\) unambiguously in the whole \( s \) plane, so that the preceding requirements are met. Let \((R_1, \theta_1)\) and \((R_2, \theta_2)\) be two sets of polar coordinates whose origins are at \( s = 1 \) and \( s = -1 \), respectively; we choose the ranges for \( \theta_1 \) and \( \theta_2 \) as \(-\pi < \theta_1 \leq \pi\) and \(-\pi < \theta_2 \leq \pi\). Any point in the plane (except \( s = \pm 1 \)) has a definite set of coordinates \( R_1, R_2 \) and \( \theta_1, \theta_2 \). We define \((s^2 - 1)^{1/2}\) unambiguously in the whole plane by the formula

\[
(s^2 - 1)^{1/2} = R_1^{1/2}R_2^{1/2}e^{i\theta_1/2}e^{i\theta_2/2},
\]

(5.162)

where \(R_1^{1/2}\) and \(R_2^{1/2}\) are the positive square roots of the positive numbers \(R_1\) and \(R_2\), respectively. Of course, if \( s = \pm 1 \), we have \((s^2 - 1)^{1/2} = 0\). The definition (5.162) is equivalent to the definition

\[
(s^2 - 1)^{1/2} = (s - 1)^{1/2}(s + 1)^{1/2},
\]

where the branches for \((s - 1)^{1/2}\) and \((s + 1)^{1/2}\) are chosen, respectively, from \(-\infty\) to 1 and from \(-\infty\) to \(-1\) on the real axis of the \( s \) plane.

The single-valued function of \( s \) defined by (5.162) is clearly analytic in the \( s \) plane with the possible exception of the real axis. Let \((s^2 - 1)^{1/2}^+\) stand for the value of (5.162) directly above and on the real axis, and let \((s^2 - 1)^{1/2}^-\) stand for the value of (5.162) directly below the real axis. We easily calculate from (5.162) that

\[
(s^2 - 1)^{1/2}^+ = \begin{cases} 
|{(s^2 - 1)^{1/2}|}, & \text{for } s > 1 \text{ (that is, for } \theta_1 = \theta_2 = 0); \\
i|{(1 - s^2)^{1/2}|}, & \text{for } -1 < s < 1 \\
& \text{(that is, for } \theta_1 = \pi, \theta_2 = 0); \\
-|{(s^2 - 1)^{1/2}|}, & \text{for } s < -1 \text{ (that is, for } \theta_1 = \pi, \theta_2 = \pi); \\
|{(s^2 - 1)^{1/2}|}, & \text{for } s > 1 \text{ (that is, for } \theta_1 = \theta_2 = 0); \\
- i|{(1 - s^2)^{1/2}|}, & \text{for } -1 < s < 1 \\
& \text{(that is, for } \theta_1 = -\pi, \theta_2 = 0); \\
-|{(s^2 - 1)^{1/2}|}, & \text{for } s < -1 \\
& \text{(that is, for } \theta_1 = -\pi, \theta_2 = -\pi).
\]

(5.163)

\[
(s^2 - 1)^{1/2}^- = \begin{cases} 
|{(s^2 - 1)^{1/2}|}, & \text{for } s > 1 \text{ (that is, for } \theta_1 = \theta_2 = 0); \\
-|{(1 - s^2)^{1/2}|}, & \text{for } -1 < s < 1 \\
& \text{(that is, for } \theta_1 = -\pi, \theta_2 = 0); \\
|{(s^2 - 1)^{1/2}|}, & \text{for } s > 1 \text{ (that is, for } \theta_1 = \theta_2 = 0); \\
- i|{(1 - s^2)^{1/2}|}, & \text{for } -1 < s < 1 \\
& \text{(that is, for } \theta_1 = -\pi, \theta_2 = 0); \\
-|{(s^2 - 1)^{1/2}|}, & \text{for } s < -1 \\
& \text{(that is, for } \theta_1 = -\pi, \theta_2 = -\pi).
\]

(5.164)

We observe that \((s^2 - 1)^{1/2}\) is discontinuous only on the portion of the real axis between \(-1\) and \(1\), which is therefore the branch cut for \((s^2 - 1)^{1/2}\).

The inversion formula for (5.161) gives

\[
C_1(x, t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{s t} e^{-{(s^2-1)^{1/2}|x|}} \frac{e^{-{(s^2-1)^{1/2}|x|}}}{2(s^2 - 1)^{1/2}} \, ds,
\]

(5.165)
where $a > 1$. To evaluate this integral for $t < |x|$, we use Cauchy's theorem on the contour in Figure 5.2. Since the integrand in (5.165) is exponentially small on the circular part of this contour and since the integrand is analytic within the contour, we conclude that $C_1 = 0$ for $t < |x|$. If $t > |x|$, we use Cauchy's theorem for the contour in Figure 5.3. The integrand is
exponentially small on the circular part of the contour and one can also show that the segments $PP'$ and $QQ'$ do not contribute as the radius $\to \infty$. The integrand is analytic except on the branch cut $1 < s < 1$. Hence the integral (5.165) is equal to the integral around the whole closed contour and this in turn is equal to the counterclockwise integral around the branch cut; hence

$$C_1(x, t) = \frac{1}{2\pi i} \int_{-1}^{1} \left\{ \frac{e^{st}e^{-(s^2-1)^{1/2}|x|}}{2(s^2-1)^{1/2}} - \frac{e^{st}e^{-(s^2-1)^{1/2}|x|}}{2(s^2-1)^{1/2}} \right\} ds,$$

so that, using (5.163) and (5.164),

$$C_1(x, t) = \frac{1}{2\pi} \int_{-1}^{1} \frac{e^{st} \cos(|x|(1-s^2)^{1/2})}{(1-s^2)^{1/2}} ds, \quad t > |x|. \quad (5.166)$$

The substitution $s = \sin \theta$ enables us to identify (5.166) with a Bessel function:

$$C_1(x, t) = \frac{1}{2} I_0(\sqrt{t^2 - |x|^2})H(t - |x|). \quad (5.167)$$

Using (5.131) we can obtain $C_n$ for arbitrary odd values of $n$. We perform the calculation for the case $n = 3$ (and $\gamma^2 = 1$); then

$$C_3(r, t) = -\frac{1}{4\pi r} \frac{\partial}{\partial r} \left[ I_0(\sqrt{r^2 - t^2})H(t - r) \right]$$

$$= \frac{1}{4\pi r} \delta(t - r) + \frac{1}{4\pi} H(t - r) \frac{I_0(\sqrt{t^2 - r^2})}{\sqrt{t^2 - r^2}}, \quad (5.168)$$

where we have used the fact that $I_0(0) = 1$. The case $n = 2$ is treated in Exercise 5.29.

Let us return to (5.159) and consider the case $\gamma^2 \neq 1$. We shall denote the solution by $C_n(r, t; \gamma)$. The change of variables $x'_i = \gamma x_i$, $t' = \gamma t$ immediately shows that

$$C_n(r, t; \gamma) = \gamma^{n-1} C_n(\gamma r, \gamma t; 1).$$

In particular,

$$C_1(r, t; \gamma) = \frac{1}{2} I_0(\gamma \sqrt{t^2 - r^2})H(t - r),$$

$$C_3(r, t; \gamma) = \frac{1}{4\pi r} \delta(t - r) + \frac{\gamma}{4\pi} H(t - r) \frac{I_0(\gamma \sqrt{t^2 - r^2})}{\sqrt{t^2 - r^2}}.$$

Taking into account the transformation which carries (5.157) into (5.158), we find that the causal fundamental solution of the dissipative wave equation is

$$e^{-\gamma t} C_n(r, t; \gamma) = \gamma^{n-1} e^{-\gamma t} C_n(\gamma r, \gamma t; 1).$$

In the one-dimensional case this becomes

$$\frac{1}{2} e^{-\gamma t} I_0(\gamma \sqrt{t^2 - r^2})H(t - r). \quad (5.169)$$
Figure 5.4 compares the causal fundamental solutions (5.150) and (5.169), which correspond to the nondissipative and dissipative cases, respectively. If a long taut string is struck a sudden blow at \( x = 0, \ t = 0 \), it will deflect according to (5.150) or (5.169) (depending on whether or not there is air resistance). The front of the disturbance travels outward with unit velocity in either case but the magnitude of the front is smaller by a factor \( e^{-\gamma t} \) when dissipation is present. Directly under the blow (that is, at \( x = 0 \)), the asymptotic formula \( I_0(\gamma t) \sim (2/\pi \gamma t)^{1/2} e^{\gamma t} \) shows that deflection is approximately \( \frac{1}{2}(2/\pi \gamma t)^{1/2} \) for large \( t \) in the dissipative case (as compared with \( \frac{1}{2} \) in the nondissipative case).

**EXERCISES**

5.29 Consider the solution of (5.159) for the case \( n = 2, \ \gamma^2 = 1 \). Then, from (5.160) and (5.131), we have

\[
\bar{C}_2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-(s^2-1)^{1/2}rt^2} (s^2 - 1)^{-1/2} \, dz.
\]

From (5.161),

\[-2 \frac{d\bar{C}_1(|x|, \ s)}{d|x|} = e^{-(s^2-1)^{1/2}|x|},\]

so that from (5.167) we have that the inverse of \( e^{-(s^2-1)^{1/2}|x|} \) is

\[I_0(\sqrt{t^2 - |x|^2}) \delta(t - |x|) + |x| H(t - |x|) \frac{I_0(\sqrt{t^2 - |x|^2})}{\sqrt{t^2 - |x|^2}}\]

Use this result and the above expression for \( \bar{C}_2 \) to calculate \( C_2 \).

5.30 If a taut string is embedded in a medium such as rubber, which provides additional restoring forces proportional to the displacement, the
Differential equation takes the form [contrast with (5.157) and (5.158)]
\[
\frac{\partial^2 u}{\partial t^2} + k^2 u - \frac{\partial^2 u}{\partial x^2} = q(x, t),
\]
where \(k^2\) is a given positive constant.

Find the causal fundamental solution by using a Laplace transform. Compare your solution with (5.150) and (5.169) (see also Figure 5.4).

5.31 The Klein-Gordon equation which occurs in quantum mechanics (and also as the three-dimensional analogue of Exercise 5.30) is
\[
\frac{\partial^2 u}{\partial t^2} + k^2 u - \nabla^2 u = q(x, t).
\]
Find the causal fundamental solution for the case \(n = 3\).

5.32 (a) Show that the function
\[
u(x, y) = \begin{cases} 1, & x \text{ and } y \text{ both } > 0, \\ 0, & \text{otherwise}, \end{cases}
\]
is a fundamental solution for \(\frac{\partial^2}{\partial x \partial y}\). Since \(\frac{\partial^2}{\partial x \partial y}\) is formally self-adjoint, it suffices to show that for each test function \(\varphi(x, y)\),
\[
\int_0^\infty \int_0^\infty \frac{\partial^2 \varphi}{\partial x \partial y} \, dx \, dy = \varphi(0, 0).
\]
Show also that the functions \(u(-x, -y), -u(x, -y), -u(-x, y)\) are all fundamental solutions.

(b) Show that the transformation \(\xi = x - t, \eta = x + t\) carries the wave equation
\[
\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = q(x, t)
\]
into the equation \(4(\partial^2 u / \partial \xi \partial \eta) = q\). Using the result of part (a), verify that \(\frac{1}{4} H(t - |x|)\) is the causal fundamental solution of the one-dimensional wave equation.

5.33 (a) We want to find the solution of
\[
\frac{\partial^2 E}{\partial x \partial y} + \gamma^2 E = \delta(x)\delta(y),
\]
which vanishes unless \(x, y \geq 0\). Take a Laplace transform on the \(x\) coordinate to show that
\[
\bar{E}(s, y) = H(y) \frac{e^{-\gamma^2 y/s}}{s}.
\]
Perform the inversion to obtain

\[ E(x, y) = J_0(2\gamma \sqrt{xy})H(x)H(y). \]

(b) By applying the transformation in Exercise 5.32(b), obtain the causal fundamental solution for the equation of telegraphy and hence for the dissipative wave equation.

5.34 Find the fundamental solution (spherically symmetric) for the iterated Laplacian \( \nabla^4 = \nabla^2 \nabla^2 \) in one, two, and three dimensions.

5.35 Find the spherically symmetric fundamental solution of

\[ \nabla^4 E - \lambda E = \delta(x), \]

in one, two, and three dimensions. Here \( \lambda \) is an arbitrary complex number.

5.36 Solve equation (5.135) in one spatial dimension by using a Laplace transform on time. Compare your solution with (5.140).

5.37 Consider the causal fundamental solution of the \( n \)-dimensional wave equation [equation (5.142) with \( c^2 = 1 \)]. Apply an \( n \)-dimensional Fourier transform over the space coordinates to obtain

\[ \frac{d^2 \hat{C}^n}{dt^2} + \|\alpha\|^2 \hat{C}^n = \delta(t), \quad \hat{C}^n = 0 \text{ for } t < 0. \]

Hence show that

\[ \hat{C}^n = \frac{\sin \|\alpha\| t}{\|\alpha\|}. \]

Invert for the cases \( n = 1 \) and \( n = 3 \) to obtain (5.150) and (5.152). If inclination and skill permit, perform the inversion for arbitrary \( n \).

5.38 Let \( E(x, t) \) be the solution of

\[ \frac{\partial^m E}{\partial t^m} + \sum_{k=1}^{m-1} a_k \frac{\partial^k E}{\partial t^k} + LE = \delta(x)\delta(t), \quad E \equiv 0 \text{ for } t < 0, \]

where the \( a_k \) are constants and \( L \) is a linear operator which depends only on the space coordinates. Show that \( E \) can also be characterized for \( t > 0 \) as the solution of the initial value problem

\[ \frac{\partial^m E}{\partial t^m} + \sum_{k=1}^{m-1} a_k \frac{\partial^k E}{\partial t^k} + LE = 0, \quad t > 0; \]

\[ E(x, 0+) = \frac{\partial E}{\partial t} (x, 0+) = \cdots = \frac{\partial^{m-2} E}{\partial t^{m-2}} (x, 0+) = 0; \]

\[ \frac{\partial^{m-1} E}{\partial t^{m-1}} (x, 0+) = \delta(x). \]
5.39 Verify that (5.128) satisfies (5.123).

5.40 Show that (5.151) satisfies (5.153) with $n = 2$.

5.9 CLASSIFICATION OF PARTIAL DIFFERENTIAL EQUATIONS

Introduction

In this section we develop a scheme for classifying partial differential equations so as to enable us to predict the kind of initial or boundary value problems that are mathematically suitable for equations of the various classes. Many of the ideas are closely related to those of Section 5.7, which dealt with the notion of generalized solutions. We begin with the formulation of the initial value problem for ordinary differential equations with a view to extension to partial differential equations.

Consider the ordinary differential equation of the first order,

$$ Lu = a_0(x) \frac{du}{dx} + a_1(x)u = f(x), \quad (5.170) $$

where $a_0$, $a_1$, and $f$ are continuous functions. We are interested in a solution $u(x)$ in an interval $a \leq x \leq b$. Let $x_0$ be a fixed point in this interval for which $u(x_0)$ is given. The point $x_0$ is called the initial point and the given value $u(x_0)$ constitutes the initial data.

We recall the existence and uniqueness theorem: If $a_0(x) \neq 0$ in $a \leq x \leq b$, there exists one and only one solution $u(x)$ of (5.170) satisfying the initial data.

It is, in fact, rather simple to construct the solution $u(x)$ from the initial data using a step-by-step procedure. Let $x_1 = x_0 + \Delta x$, where $\Delta x$ is small; then

$$ u(x_1) = u(x_0) + \frac{du}{dx}(x_0)\Delta x. \quad (5.171) $$

Now $u(x_0)$ is given and $du(x_0)/dx$ can be calculated from the differential equation

$$ a_0(x_0)\frac{du}{dx}(x_0) = -a_1(x_0)u(x_0) + f(x_0). \quad (5.172) $$

Thus, since $a_0(x_0) \neq 0$,

$$ \frac{du}{dx}(x_0) = \frac{-a_1(x_0)}{a_0(x_0)} u(x_0) + \frac{f(x_0)}{a_0(x_0)}, \quad (5.173) $$

and (5.171) enables us to find $u(x_1)$. Considering $x_1$ as a new initial point with initial data $u(x_1)$ we can next calculate $u(x_1 + \Delta x) = u(x_2)$ and proceed in this way to determine $u(x)$ in the entire interval $a \leq x \leq b$.
The procedure just described depends on our ability to calculate \( du(x_0)/dx \) in an unambiguous manner. This can be done if \( a_0(x_0) \neq 0 \). If \( a_0(x_0) = 0 \), we can no longer pass from (5.172) to (5.173); from (5.172) we note that either \( du(x_0)/dx \) does not exist [if the given quantity \(-a_1(x_0)u(x_0) + f(x_0) \neq 0\)], or else \( du(x_0)/dx \) is indeterminate [if \(-a_1(x_0)u(x_0) + f(x_0) = 0\)]. In any event, if \( a_0(x_0) = 0 \), the initial data together with the differential equation do not suffice to calculate \( du(x_0)/dx \) unambiguously. The difficulty which occurs when \( a_0(x_0) = 0 \) can be stated in another way: On the one hand, the value of \( Lu \) at \( x_0 \) is known from the differential equation, that is, \( Lu = f(x_0) \); on the other hand, since \( a_0(x_0) = 0 \), \( Lu \) at \( x_0 \) is completely determined from the initial data, that is, \( Lu = a_1(x_0)u(x_0) \). Thus whenever at a point \( x_0 \) the value of \( Lu \) can be calculated from the initial data alone, we can expect that the existence and uniqueness theorem will be violated.

A similar situation occurs for a linear differential equation of order \( p \). If the leading coefficient \( a_0(x) \) does not vanish, the given initial data \( u(x_0), \ldots, u^{(p-1)}(x_0) \) can be substituted in the differential equation \( Lu = f \) to calculate unambiguously \( u^{(p)}(x_0) \). Then in turn we can compute

\[
 u(x_1), \ldots, u^{(p-1)}(x_1);
\]

the last of these from \( u^{(p-1)}(x_1) = u^{(p-1)}(x_0) + u^{(p)}(x_0) \Delta x \). With \( x_1 \) as a new starting point and the new initial data we can calculate \( u(x_2) \), etc. If \( a_0(x_0) \) vanishes, then \( u^{(p)}(x_0) \) cannot be found from \( Lu = f \) and the initial data, so that the solution \( u(x) \) cannot be constructed; alternatively, \( Lu \) at \( x_0 \) can be determined independently either from the initial data or from the differential equation, so that the existence and uniqueness theorem cannot be expected to hold.

**Cauchy Problem (or Initial Value Problem)**

Consider a linear partial differential equation of order \( p \) in the \( n \) variables \( x_1, \ldots, x_n \):

\[
 Lu = f(x),
\]

where

\[
 L = \sum_{|k| \leq p} a_k(x) D^k.
\]

The given function \( f(x) \) and the given set of functions \( \{a_k(x)\} \) are assumed continuous; further restrictions will occasionally be made on these functions in deriving some particular results. A solution \( u(x) \) is sought in a specified region \( R \) in \( R_n \).

We shall associate certain initial data with the differential equation (5.174). The most natural extension of the one-dimensional case would require the specification of \( u \) and all its derivatives of order \( \leq p - 1 \) (with respect to \( x_1, \ldots, x_n \)) on a smooth hypersurface \( \sigma \) of dimension \( n - 1 \). It is equally
effective and considerably more economical to specify only the so-called Cauchy data on $\sigma$, that is, the value of $u$ and its normal derivatives of order $\leq p - 1$. We now show that the Cauchy data on $\sigma$ enables us to calculate on $\sigma$ all derivatives of order $\leq p - 1$ with respect to $x_1, \ldots, x_n$.

Let $P$ be a point on $\sigma$; by projecting $\sigma$ on the tangent plane to $\sigma$ at $P$, we establish a one-to-one correspondence between points on the tangent plane and points on $\sigma$ (at least for a sufficiently small neighborhood of $P$). On the tangent plane we construct an $(n - 1)$-dimensional Cartesian coordinate system, with origin at $P$. An $(n - 1)$-dimensional coordinate system is therefore induced on $\sigma$ by assigning to each point on $\sigma$ the corresponding values of the $n - 1$ Cartesian coordinates of the projection of the point on the tangent plane. These $n - 1$ "tangential" coordinates will be labeled $\xi_2, \ldots, \xi_n$. If $\lambda$ stands for a coordinate normal to $\sigma$, then $(\lambda, \xi_2, \ldots, \xi_n)$ is a coordinate system for all points in $R_n$ sufficiently close to $P$. In this new coordinate system, we observe that the point $P$ has coordinates $(0, \ldots, 0)$.

The Cauchy data on $\sigma$ entails a knowledge of

$$u(0, \xi_2, \ldots, \xi_n), \frac{\partial u}{\partial \lambda}(0, \xi_2, \ldots, \xi_n), \ldots, \frac{\partial^{p-1} u}{\partial \lambda^{p-1}}(0, \xi_2, \ldots, \xi_n).$$

From $u(0, \xi_2, \ldots, \xi_n)$, partial derivatives of all orders with respect to $\xi_2, \ldots, \xi_n$ are known on $\sigma$ for a neighborhood of $P$. Any such derivative is labeled a tangential derivative. Since the normal derivatives of order $\leq p - 1$ are given on $\sigma$, we actually know on $\sigma$ all derivatives of order $\leq p - 1$ with respect to $\lambda, \xi_2, \ldots, \xi_n$. By using the transformation law connecting $(\lambda, \xi_2, \ldots, \xi_n)$ and $(x_1, \ldots, x_n)$ we can therefore calculate on $\sigma$ all derivatives of $u$ of order $\leq p - 1$ with respect to $x_1, \ldots, x_n$.

One should also observe that some derivatives of order greater than $p - 1$ can be calculated on $\sigma$ from the Cauchy data. For instance, any tangential derivative of any order is known on $\sigma$; moreover, such $p$th-order derivatives as

$$\frac{\partial}{\partial \xi_2} \frac{\partial^{p-1} u}{\partial \lambda^{p-1}}, \ldots,$$

are known on $\sigma$. In terms of the coordinates $\lambda, \xi_2, \ldots, \xi_n$, the only derivative order $p$ which cannot be calculated on $\sigma$ from the Cauchy data alone is $\partial^p u/\partial \lambda^p$.

**Definition.** Any differential operator $L$ such that $Lu$ can be evaluated at a point $P$ on $\sigma$ from the Cauchy data alone is known as an interior differential operator at $P$ with respect to $\sigma$; and the surface $\sigma$ is termed characteristic for $L$ at $P$. If the surface $\sigma$ is characteristic for $L$ at every point $P$, we say that $\sigma$ is a characteristic surface for $L$.

**Remark.** When there is no danger of confusion, the various qualifiers "for $L$," "for $L$ at $P$," "at $P$ with respect to $\sigma$," may be omitted.
EXAMPLE

Consider the first-order operator $\partial / \partial x_1$ in $R_2$. The hypersurface $\sigma$ is now a curve $C$ and the Cauchy data consists of giving $u$ on $C$. Under what circumstances does the Cauchy data supply the value of $\partial u / \partial x_1$ at a point $P$ on $C$? Clearly it is necessary and sufficient for the tangent line to $C$ at $P$ to be in the $x_1$ direction. Therefore $\partial / \partial x_1$ is an interior differential operator at $P$ with respect to $C$ if and only if the tangent line to $C$ at $P$ is in the $x_1$ direction; if this happens at the point $P$, $C$ is characteristic for $\partial / \partial x_1$ at $P$. The characteristic curves for $\partial / \partial x_1$ are the straight lines $x_2 = \text{constant}$.

The notions of "interior differential operator" and "characteristic" are important because they reflect the same difficulty in $n$ dimensions which occurred when the coefficient of the highest derivative vanished for the one-dimensional problem.

Theorem 1. The following properties are all equivalent:
1. $L$ is an interior differential operator with respect to $\sigma$ at $P$.
2. $\sigma$ is characteristic for $L$ at $P$.
3. In the coordinate system $(\lambda, \xi_2, \ldots, \xi_n)$, the coefficient of $\partial^p / \partial \lambda^p$ in $L$ vanishes at $P$.

Proof. Properties (1) and (2) are equivalent by definition. We show that (1) and (3) are equivalent. Let $L = \sum_{|k| \leq p} a_k(x) D^k$ and $\lambda = \xi_1, \xi_2, \ldots, \xi_n$ be the normal and tangential coordinates previously introduced for a neighborhood of $P$. In this coordinate system the coordinates of $P$ all vanish, and $L$ has the form $\sum_{|k| \leq p} b_k(\xi) D^k$, where

$$D^k_\xi = \frac{\partial^{|k|}}{\partial \xi_1^{k_1} \cdots \partial \xi_n^{k_n}}$$

and $b_k(\xi) = b_k(\xi_1, \ldots, \xi_n)$.

Of course, it may be difficult to calculate the new coefficients $\{b_k(\xi)\}$, but this calculation is not required for the present purposes. Setting

$$b(\xi) = b_{p,0,\ldots,0}(\xi),$$

we have

$$L = b(\xi) \frac{\partial^p}{\partial \lambda^p} + L', \quad (5.175)$$

where $L'$ is an interior differential operator at $P$ with respect to $\sigma$. (Since $L'$ involves only normal derivatives of order $\leq p - 1$ and tangential derivatives, $L'$ can be calculated at $P$ from the Cauchy data on $\sigma$.) If $b(P) = b(0, \ldots, 0)$ vanishes, then $L$ is itself interior at $P$ with respect to $\sigma$. If $L$ is interior, then, since $\partial^p u / \partial \lambda^p$ is not known from the Cauchy data, $b(\xi)$ must vanish at $P$. 


Consider now the Cauchy problem
\[ Lu = f, \] (5.176)
with Cauchy data on \( \sigma \). Let \( P \) be a point on \( \sigma \).

**Theorem 2.** If \( \sigma \) is characteristic for \( L \) at \( P \), the differential equation \( Lu = f \) and the Cauchy data do not suffice to compute \( \partial^p u/\partial \xi^p \) at \( P \).

**Proof.** From (5.175), we have
\[ Lu = b(\xi) \frac{\partial^p u}{\partial \xi^p} + Lu = f(\xi), \] (5.177)
where \( L' u \) is known from the Cauchy data. Since \( b(0, \ldots, 0) = 0 \), either \( \partial^p u/\partial \xi^p(0, \ldots, 0) \) does not exist (if \( f - L' u \) does not vanish at \( \xi = 0 \)) or \( \partial^p u/\partial \xi^p(0, \ldots, 0) \) is indeterminate (if \( f - L' u \) vanishes at \( \xi = 0 \)).

**Theorem 3.** If \( \sigma \) is not characteristic for \( L \) at \( P \), \( \partial^p u/\partial \xi^p \) is unambiguously determined at \( P \) from the differential equation and the Cauchy data.

**Proof.** Immediate from (5.177) by dividing by \( b(0, \ldots, 0) \).

**Theorem 4.** All \( p \)th-order derivatives of \( u \) with respect to \( x_1, \ldots, x_n \) are unambiguously determined at \( P \) from the Cauchy data on \( \sigma \) and the differential equation if and only if \( \sigma \) is not characteristic for \( L \) at \( P \).

The gist of these theorems is that if \( \sigma \) is characteristic for \( L \) at \( P \), there is no hope for uniqueness and existence of the solution of the Cauchy problem (5.176) (not even in a neighborhood of \( P \)). Of course the content of these theorems is essentially negative. One might hope that, if \( \sigma \) is not characteristic for \( L \) at \( P \), the solution of the Cauchy problem exists and is uniquely determined in at least a neighborhood of \( P \). Unfortunately, this hope is not fulfilled; even if \( \sigma \) is nowhere characteristic for \( L \), the Cauchy problem may not have a solution (see the discussion on elliptic equations). At this stage we are forced to abandon the general theory and study some particularly simple (but also particularly useful) special cases.

**Partial Differential Equations of the First Order**

We consider problems in the two independent variables \( x \) and \( y \). The nature of the difficulties that arise is illustrated by the simple Cauchy problem
\[ \frac{\partial u}{\partial x} = 0, \quad u \text{ given on the smooth curve } C. \] (5.178)

We have already established that \( C \) is characteristic at a point \( P \) for \( \partial/\partial x \) if and only if the tangent to \( C \) at \( P \) is in the \( x \) direction. The characteristic curves are the straight lines \( y = \text{constant} \). Before proceeding to the discussion of the solution of (5.178) for a variety of initial curves \( C \), we observe that the
general solution of the differential equation $\partial u/\partial x = 0$ is $u = F(y)$, where $F$ is an arbitrary function.

**Case 1.** The curve $C$ is a part or the whole of the line $x = x_0$ (see Figure 5.5). Thus we give $u(x_0, y)$ in $y_1 \leq y \leq y_2$. The curve $C$ is nowhere characteristic.

![Figure 5.5](image-url)

**FIGURE 5.5**

The solution of (5.178) is uniquely determined in the shaded region and $u(x, y) = u(x_0, y)$. To obtain the solution at point II, for instance, we merely pick off the given value of $u$ at point I.

**Case 2.** The smooth curve $C$ is an arbitrary curve which is nowhere characteristic, as in Figure 5.6.

![Figure 5.6](image-url)

**FIGURE 5.6**

From the given values of $u$ on $C$, the solution is uniquely determined in the shaded region and $u_{II} = u_I$. There is no information on the solution $u$ outside the shaded region.

**Remark.** Note that the given values of $u$ do not have to be continuous. For instance, if, on the curve $C$, $u$ is discontinuous at $I$, then the solution will be perfectly well defined in the shaded region, but will be discontinuous on the entire horizontal line through $I$. We say that discontinuities in initial data propagate along horizontal lines (this property of characteristics was discussed in Section 5.7).
Case 3. $C$ is a segment of a horizontal line (see Figure 5.7), so that $C$ is a portion of a characteristic curve.

If $u$ is given arbitrarily on this segment, these values will contradict the differential equation $\partial u/\partial x = 0$ and no solution is possible. If, by accident, the given values of $u$ are constant on the segment, the solution is undetermined except on the extension of the segment. Even if $C$ is characteristic only at one point $P$ (see Figure 5.8) we run into difficulties.

In fact, the data on $C$ are again incompatible with the differential equation, unless the given values $u_{II}$ and $u_I$ happen to coincide. Since $\partial u/\partial x = 0$, $u$ must be constant on the line $y = y_1$, so that $u_{III} = u_{II} = u_I$, and, unless $u_{II} = u_I$, no solution is possible.

We are now ready to discuss the Cauchy problem for the general linear equation of the first order:

\[ a(x, y) \frac{\partial u}{\partial x} + b(x, y) \frac{\partial u}{\partial y} + c(x, y)u + d(x, y) = 0. \]  

(5.179)

The initial data (that is, the value of $u$) are given on a smooth curve $C$. Let $P$ be a point on $C$; if we can calculate $\partial u/\partial x$ and $\partial u/\partial y$ unambiguously at $P$ from the data and the differential equation, then and only then will $C$ not be characteristic at $P$ (by Theorem 4). Consider two neighboring points
\( P = (x, y) \) and \( Q = (x + dx, y + dy) \) located on the initial curve. Let us try to calculate \( \partial u/\partial x \) and \( \partial u/\partial y \) at \( P \). The following relations must hold:

\[
\frac{u(Q) - u(P)}{dx} = \frac{\partial u}{\partial x}(P)dx + \frac{\partial u}{\partial y}(P)dy,
\]

\[
-c(P)u(P) - d(P) = a(P)\frac{\partial u}{\partial x}(P) + b(P)\frac{\partial u}{\partial y}(P).
\]

These simultaneous linear equations will determine \( \partial u(P)/\partial x \) and \( \partial u(P)/\partial y \) unambiguously if and only if

\[
\begin{vmatrix}
    dx & dy \\
    a(P) & b(P)
\end{vmatrix} \neq 0 \quad \text{or} \quad b(P)dx - a(P)dy \neq 0. \tag{5.180}
\]

The curve \( C \) is characteristic at \( P = (x, y) \) if and only if

\[
b(x, y)dx - a(x, y)dy = 0. \tag{5.181}
\]

Thus at every point in the plane a unique characteristic direction is defined by (5.181) (we have assumed that \( a \) and \( b \) do not both vanish at the same point). If the ordinary differential equation (5.181) is integrated, a "one-parameter" family of curves is obtained; these are the characteristic curves for (5.179).

We state without proof a theorem similar to the one obtained for the equation \( \partial u/\partial x = 0 \): If \( C \) is nowhere characteristic, the solution of (5.179) exists and is unique in the characteristic "strip" bounded by the characteristic curves which pass through the end points of \( C \). If the initial data have a discontinuity at a point \( A \) on \( C \), the discontinuity is propagated along the characteristic curve through \( A \). The solution is then a generalized solution in the sense of Section 5.7 (see also Exercise 5.24).

If the curve \( C \) is characteristic at even a single point \( P \), the initial data may be incompatible with the differential equation.

**Partial Differential Equations of the Second Order**

Consider the linear equation of the second order in two independent variables,

\[
a(x, y) \frac{\partial^2 u}{\partial x^2} + 2b(x, y) \frac{\partial^2 u}{\partial x \partial y} + c(x, y) \frac{\partial^2 u}{\partial y^2} + d\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0, \tag{5.182}
\]

where

\[
d\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = e(x, y) \frac{\partial u}{\partial x} + f(x, y) \frac{\partial u}{\partial y} + g(x, y)u + h(x, y).
\]

We are given \( u \) and its normal derivative on a smooth curve \( C \). Thus both \( \partial u/\partial x \) and \( \partial u/\partial y \) are known on \( C \).
The curve $C$ will not be characteristic at $P$, if the second derivatives $\partial^2 u/\partial x^2$, $\partial^2 u/\partial x \partial y$, and $\partial^2 u/\partial y^2$ can be calculated unambiguously from the data and the differential equation. Let $P = (x, y)$ and $Q = (x + dx, y + dy)$ be two neighboring points on $C$. Then

$$\frac{\partial u}{\partial x} (Q) - \frac{\partial u}{\partial x} (P) = \frac{\partial^2 u}{\partial x^2} (P) dx + \frac{\partial^2 u}{\partial x \partial y} (P) dy,$$

$$\frac{\partial u}{\partial y} (Q) - \frac{\partial u}{\partial y} (P) = \frac{\partial^2 u}{\partial x \partial y} (P) dx + \frac{\partial^2 u}{\partial y^2} (P) dy,$$

$$-d \begin{bmatrix} P, u(P), \frac{\partial u}{\partial x} (P), \frac{\partial u}{\partial y} (P) \end{bmatrix} = a(P) \frac{\partial^2 u}{\partial x^2} (P) + 2b(P) \frac{\partial^2 u}{\partial x \partial y} (P) + c(P) \frac{\partial^2 u}{\partial y^2} (P).$$

The left sides of these equations are known from the Cauchy data. Hence we have three simultaneous linear algebraic equations for $\partial^2 u(P)/\partial x^2$, $\partial^2 u(P)/\partial x \partial y$, and $\partial^2 u(P)/\partial y^2$. The necessary and sufficient condition for a unique solution is

$$\begin{vmatrix} dx & dy & 0 \\ 0 & dx & dy \\ a(P) & 2b(P) & c(P) \end{vmatrix} \neq 0 \quad \text{or} \quad a \, dy^2 - 2b \, dx \, dy + c \, dx^2 \neq 0.$$  (5.183)

If $a(P) \neq 0$, $C$ will be characteristic at $P = (x, y)$ if and only if

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - ac}}{a}. \quad (5.184)$$

We distinguish among three cases:

1. If $b^2 - ac < 0$, no real curve $C$ can satisfy (5.184) and no curve $C$ is characteristic at $P$. The differential equation (5.182) is said to be elliptic.

2. If $b^2 - ac > 0$, there are two characteristic directions at $P$. The differential equation (5.182) is called hyperbolic.

3. If $b^2 - ac = 0$, there is only one characteristic direction at $P$, and (5.182) is parabolic.

**Remarks.** 1. It should be emphasized that this is a local classification. The equation

$$\frac{\partial^2 u}{\partial x^2} + x \frac{\partial^2 u}{\partial y^2} = 0$$

is hyperbolic for $x < 0$ and elliptic for $x > 0$.

2. If the coefficients $a, b,$ and $c$ are constants, the equation is of the same type in the whole plane.

3. The classification depends only on the coefficients of the derivatives of the highest order appearing in (5.182).
In the hyperbolic case, we can find two families of characteristic curves (that is, curves characteristic at every point) by integrating the ordinary differential equation (5.184). In the parabolic case there exists a single family of characteristic curves.

**EXAMPLES**

The Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is elliptic, as is the Helmholtz equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k^2 u = 0.$$  

The wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

is *hyperbolic* (here the y coordinate is the time, so we have relabeled it t; $c^2$ is a given constant). The characteristic curves are the straight lines $x + ct = \text{constant}$ and $x - ct = \text{constant}$.

The equation $\frac{\partial^2 u}{\partial x \partial y} = 0$ is *hyperbolic* and the characteristic curves are the straight lines $x = \text{constant}$, $y = \text{constant}$.

The diffusion equation (or equation of heat conduction)

$$\frac{\partial u}{\partial t} - a \frac{\partial^2 u}{\partial x^2} = 0$$

is *parabolic*. Here $t$ is the time variable and $a$ is a given positive constant. The characteristic curves consist of the single family $t = \text{constant}$.

At first glance one might not suspect the great differences in these equations. A mere change of sign in one of the terms distinguishes the Laplace equation from the wave equation (with $c = 1$), yet the physical problems and, fortunately, the corresponding mathematical formulations are completely different for the two equations. Since the Cauchy problem in its original form turns out to be particularly suitable for hyperbolic equations, we begin with a few examples of this kind.

**Hyperbolic Equation** $\frac{\partial^2 u}{\partial x \partial y} = 0$

Since $\frac{\partial}{\partial x}(\frac{\partial u}{\partial y}) = 0$, $\frac{\partial u}{\partial y} = G(y)$ and $u = F(x) + H(y)$, which is the general solution of $\frac{\partial^2 u}{\partial x \partial y} = 0$. The characteristics are the straight lines $x = \text{constant}$, $y = \text{constant}$. Let us examine a few initial data problems for this equation.
1. The Cauchy problem. Let $C$ be a curve which is nowhere characteristic (as in Figure 5.9). We shall find that the solution exists and is uniquely determined in the characteristic quadrangle generated by $C$ (the shaded area in the figure).

The solution at $P$ is easily found in terms of the data on $C$. We have

$$u(P) = u(Q) + \int_{Q}^{P} \frac{\partial u}{\partial x}(\xi, y)d\xi;$$

from the differential equation, $\partial/\partial y(\partial u/\partial x) = 0$; hence $\partial u/\partial x$ is a function of $x$ only and the integral from $Q$ to $P$ can just as well be taken along $C$. If the equation of $C$ is $y = \psi(x)$ or $x = \psi^{-1}(y)$,

$$u(P) = u(Q) + \int_{Q}^{R} \frac{\partial u}{\partial x} [\xi, \psi(\xi)]d\xi$$

or

$$u(x, y) = u[\psi^{-1}(y), y] + \int_{\psi^{-1}(y)}^{x} \frac{\partial u}{\partial x} [\xi, \psi(\xi)]d\xi.$$

Since $u$ and $\partial u/\partial x$ are known on $C$, the above formula is the desired answer. Alternatively, the solution can be expressed in terms of $\partial u/\partial n$. Note that

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial s} \cos \theta + \frac{\partial u}{\partial n} \sin \theta$$

and $dx = ds \cos \theta$;

hence

$$u(P) = u(Q) + \int_{Q}^{R} \left( \frac{\partial u}{\partial s} \cos^2 \theta + \frac{\partial u}{\partial n} \sin \theta \cos \theta \right)ds. \quad (5.185)$$
In the special but important case where $C$ is a $45^\circ$ line, $\cos \theta = \sin \theta = \sqrt{2}/2$ and,

$$u(P) = u(Q) + \int_{Q}^{R} \left( \frac{1}{2} \frac{\partial u}{\partial s} + \frac{1}{2} \frac{\partial u}{\partial n} \right) ds = \frac{u(Q)}{2} + \frac{u(R)}{2} + \frac{1}{2} \int_{Q}^{R} \frac{\partial u}{\partial n} ds. \quad (5.186)$$

2. If $C$ is characteristic at $P$, the Cauchy data are generally incompatible with the differential equation. This is clearly seen if $C$ is a portion of the characteristic curve $y = \text{constant}$. The Cauchy data then consist of arbitrarily assigning $u$ and $\partial u/\partial y$ on $C$; but, from the differential equation, $\partial u/\partial y$ is independent of $x$, so that $\partial u/\partial y$ must be constant on $C$.

3. There are sensible problems which can be formulated with data on characteristics, but of course these problems cannot be strict Cauchy problems. Consider the two intersecting characteristic segments $OA$ and $OB$, which, for simplicity, we shall take along the coordinate axes as in Figure 5.10. We show that if $u$ alone is given on $OA$ and $OB$, there exists a unique solution of $\partial^2 u/\partial x \partial y = 0$ in the shaded quadrangle.

![Figure 5.10](image)

The given data are $u(x, 0) = f(x)$ on $OA$ and $u(0, y) = g(y)$ on $OB$. Since the general solution of the differential equation is

$$u(x, y) = F(x) + G(y),$$

we must determine $F$ and $G$ from $f$ and $g$. We have

$$u(x, 0) = F(x) + G(0) = f(x),$$

$$u(0, y) = F(0) + G(y) = g(y),$$

so that

$$u(x, y) = f(x) + g(y) - F(0) - G(0).$$
Assuming that the given values of \( f(x) \) and \( g(y) \) coincide at the origin, we have
\[
f(0) = g(0) = u(0, 0);
\]
hence
\[
u(P) = u(x, y) = f(x) + g(y) - u(0, 0) = u(R) + u(Q) - u(O). \quad (5.187)
\]

It is now easy to check that (5.187) actually furnishes the solution of our problem. We note that this solution depends continuously on the initial data; that is, if \( f \) and \( g \) are changed slightly, the change in the solution \( u \) is small. Further, if \( f(x) \) has a discontinuity, say at \( x = x_0 \), the discontinuity is "propagated" along the characteristic \( x = x_0 \). Strictly speaking, the classical theory allows only discontinuities in the second or higher derivatives, since otherwise we cannot perform the differentiations required by the differential equation. However, from the point of view of generalized solutions, discontinuities in \( f \) and in \( df/dx \) are permitted. It is therefore possible for a discontinuous function to be a solution of a homogeneous equation. This phenomenon cannot take place for parabolic and elliptic equations, where discontinuities in the data are smoothed out as soon as we leave the initial curve \( C \).

4. Another type of problem occurs when \( u \) is given on a characteristic curve and on an intersecting curve nowhere characteristic, as in Figure 5.11. We have
\[
u(x, y) = \int_R^P \frac{\partial u}{\partial x} (\xi, y) d\xi + u(R) = u(R) + \int_Q^O \frac{\partial u}{\partial x} (\xi, y) d\xi
\]
or
\[
u(P) = u(R) + u(Q) - u(O). \quad (5.188)
\]
This problem therefore has a unique solution for arbitrary data.

**Hyperbolic Equation** \( (\partial^2 u / \partial t^2) - c^2 (\partial^2 u / \partial x^2) = 0 \)

This is the wave equation with \( x \), a space coordinate, and \( t \), the time coordinate. This equation governs the deflection \( u \) of a vibrating string. The transformation \( \xi = x - ct \), \( \eta = x + ct \), reduces the wave equation to
\[ \frac{\partial^2 u}{\partial \xi \partial \eta} = 0. \] The characteristics of the wave equation are the lines \( x - ct = \) constant, \( x + ct = \) constant, and under the transformation of variables they become the characteristics for the equation \( \frac{\partial^2 u}{\partial \xi \partial \eta} = 0. \) This is an example of a general property of hyperbolic equations: under a sufficiently smooth transformation, characteristics are mapped into characteristics.

One can obtain all results for the wave equation by reducing it to the problems of the previous example. We content ourselves with giving the answers to some of the important questions.

1. If \( u \) and \( \frac{\partial u}{\partial t} \) are given at \( t = 0 \), we have the simplest and most important Cauchy problem (note that \( t = 0 \) is not a characteristic). Although the solution for \( t < 0 \) can be found, the interest lies in the solution for \( t > 0. \) Physically, \( u(x, t) \) may represent, for instance, the deflection of an infinitely long string whose initial displacement and velocity are given.

As in the special case of the first problem considered for the equation \( \frac{\partial^2 u}{\partial x \partial y} = 0, \) the solution at \( P \) is

\[
 u(x, t) = u(P) = \frac{u(Q) + u(R)}{2} + \frac{1}{2c} \int_Q^R \frac{\partial u}{\partial t} \, dx,
\]

where reference is made to Figure 5.12. If the initial value of \( u(x, t) \) is \( f(x), \) and the initial value of \( \frac{\partial u}{\partial t} \) is \( g(x), \) we have

\[
 u(x, t) = \frac{f(x - ct) + f(x + ct)}{2} + \frac{1}{2c} \int_{x - ct}^{x + ct} g(\xi) \, d\xi,
\]  

(5.189)

which is known as d'Alembert's formula. The properties of the solution will be discussed in detail in Chapter 7.

\[ \text{FIGURE 5.12} \]
2. Another typical problem for the wave equation involves both initial and boundary data. As an example, consider the free vibrations of a string whose ends $x = 0$ and $x = l$ are kept fixed. The deflection $u(x, t)$ satisfies the equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < x < l, \quad t > 0;$$

the initial conditions

$$u(x, 0) = f(x); \quad \frac{\partial u}{\partial t}(x, 0) = g(x), \quad 0 < x < l;$$

and the boundary conditions

$$u(0, t) = u(l, t) = 0, \quad t > 0.$$

The solution can be obtained by a variety of elementary methods (see Chapter 7), but here we shall use the method of characteristics. Referring to Figure 5.13, we see that the initial data determine the solution in region I from (5.189). Next we proceed to construct the solution in regions II and III. In region II, we note that $u = 0$ on the noncharacteristic $x = 0$, and, moreover, we claim that $u$ is also known on the side of $AC$ in region II. In fact, if $\Delta u$ stands for the jump in $u$ across $AC$ from region I to region II, then $\Delta u$ is constant [see (5.92)]. Further, $\Delta u$ is known at $A$ and we have $\Delta u = 0 - f(0+)$.
Thus in region II we have a problem in which $u$ is known on a characteristic and on an intersecting curve nowhere characteristic. The solution in region II can then be obtained as in (5.188). A similar argument holds for region III. By using the fact that the jump in $u$ across $CE$ and across $CD$ are constant, we know the value of $u$ on $CE$ and $CD$ just inside region IV. We are then faced with a problem for region IV similar to the one whose solution was given in (5.187). After having found the solution in IV, we can proceed to successive regions by considerations of the same kind.
6.1 INTRODUCTION

The mathematical formulation of physical problems often leads to a partial differential equation for the quantity of physical interest, with supplementary data provided in the form of boundary conditions (the term is used here in its general sense to include initial conditions if time is one of the independent variables). Such a combination of a partial differential equation and boundary conditions is known as a boundary value problem.

If the physical situation is deterministic, the mathematical formulation should reflect this fact and it is necessary to demand that the boundary value problem have one and only one solution. Mathematical questions of existence and uniqueness can therefore help us in deciding whether or not we have used a sensible mathematical model of the physical problem. Another, more subtle requirement must often be imposed on the boundary value problem as is illustrated by the following example. Let \( u(x) \) be the steady-state temperature in a bounded, open region \( R \) under the influence of sources which generate heat at the given rate \( q(x) \) per unit volume per unit time. In addition, the temperature is prescribed on the boundary \( \sigma \) as the function \( f(x) \). Then \( u(x) \) is the solution of the boundary value problem

\[-\nabla^2 u = q(x), \quad x \text{ in } R; \quad u = f(x), \quad x \text{ on } \sigma,\]

where, with no loss of generality, we have set the thermal conductivity equal to 1. The solution \( u \) depends on the given functions \( q \) and \( f \), and they, in turn, are not known precisely, since they are usually determined by experiment. In our particular physical problem we expect that small observational errors in \( q \) and \( f \) should not appreciably change the predicted temperature \( u \). We should
therefore require that $u$ depend continuously on the source term $q$ and the boundary data $f$.

We are thus led to the following definition of a well-posed boundary value problem:

1. There exists a solution of the problem.
2. This solution is unique.
3. The solution depends continuously on the source term and on the boundary data.†

Occasionally a physical phenomenon is described by a boundary value problem that is not well posed, but in such cases there are very serious implications as to the instability of the physical problem. A thorough investigation is then needed to decide whether it is the physical problem which is in some sense unstable or whether, instead, an error has been made in translating the physical problem into its mathematical formulation.

One of the goals of this chapter is to show that a number of boundary value problems for the Laplace equation $\nabla^2 u = 0$ and for the Poisson equation $-\nabla^2 u = q$ are well posed. Another major goal is to devise methods for the explicit construction of the solutions of these boundary value problems. We shall repeatedly make use of the divergence theorem and of the two Green's theorems:

**Divergence theorem:**

$$\int_R \text{div} \ A \ dx = \int_\sigma A \cdot n \ dS, \quad (6.1)$$

**First Green's theorem:**

$$\int_R u\nabla^2 v \ dx = -\int_R \text{grad} \ u \cdot \text{grad} \ v \ dx + \int_\sigma u \ \frac{\partial v}{\partial n} \ dS, \quad (6.1a)$$

**Second Green's theorem:**

$$\int_R (u\nabla^2 v - v\nabla^2 u) \ dx = \int_\sigma \left( u \ \frac{\partial v}{\partial n} - v \ \frac{\partial u}{\partial n} \right) dS. \quad (6.1b)$$

These formulas are valid for a wide class of bounded regions $R$ (so-called normal regions), which are the only ones that will concern us. In addition, we shall usually suppose that $A$ is a continuously differentiable vector field and that $u$ and $v$ have partial derivatives of the second order which are continuous in $R$. By appealing to the theory of distributions, one can justify the application of these equations to a much larger class of functions and we shall feel free to avail ourselves of this fact on occasion.

† Of course this criterion will have to be stated mathematically, and we shall see, through examples, that various reasonable formulations are possible.
6.2 INTERIOR DIRICHLET PROBLEM FOR THE UNIT CIRCLE

Formulation of the Problem

We consider Laplace's equation $\nabla^2 u = 0$ within the unit circle $D$ consisting of the points in the plane whose Cartesian coordinates $x_1$ and $x_2$ satisfy $x_1^2 + x_2^2 < 1$. A function which satisfies Laplace's equation is said to be harmonic. In the Dirichlet problem the supplemental data consist of giving the value of the harmonic function $u$ on the boundary $\sigma$ of $D$, that is, on the set of points $x_1^2 + x_2^2 = 1$. Introducing the polar coordinates $r$ and $\varphi$ of Figure 6.1,

we can formulate the problem as

$$\nabla^2 u = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \varphi^2} = 0 \quad \text{in } D; \quad u(1, \varphi) = f(\varphi), \quad (6.2)$$

where $f(\varphi)$ is a given, arbitrary function. We may regard $u(r, \varphi)$ as the steady-state temperature in a thin circular plate whose faces are insulated and whose edge $r = 1$ is kept at the prescribed temperature $f(\varphi)$. Alternatively, we may view $u(r, \varphi)$ as the steady temperature in an infinite circular cylinder whose boundary temperature is independent of the axial direction $z$.

The statement of the boundary condition requires some clarification to eliminate "solutions" which are physically and mathematically undesirable. For instance, the function

$$u(r, \varphi) = \begin{cases} 0, & r < 1, \\ f(\varphi), & r = 1, \end{cases} \quad (6.3)$$

satisfies (6.2) but is not the desired solution. What we really want is a solution $u(r, \varphi)$ of Laplace's equation which approaches the boundary temperature $f(\varphi)$ as we approach the boundary from the interior of the unit circle. We
must therefore interpret the boundary condition in (6.2) so as to exclude spurious "solutions" such as (6.3). Let us first discuss the case of a continuous boundary temperature; that is, \( f(\phi) \) is continuous in \(-\pi < \phi < \pi\) and \( f(\pi-) = f(-\pi+) \). This last condition will be abbreviated by writing \( f(\pi) = f(-\pi) \). A suitable formulation of the Dirichlet problem then turns out to be

\[
\nabla^2 u = 0, \quad r < 1; \\
u \text{ continuous in the closed domain } r \leq 1; \\
u(1, \phi) = f(\phi).
\]

(6.4)

Note that this formulation guarantees that, as we approach the boundary point \((1, \phi)\) along any curve whatever, \( u \) will always approach \( f(\phi) \). Exercise 6.9(b) shows that it is not sufficient to require merely that the interior temperature approach the boundary temperature along radial lines alone.

The formulation (6.4) is obviously inappropriate if the given boundary temperature is discontinuous. Suppose that we know only that \( f(\phi) \) belongs to \( \mathcal{L}_2(-\pi, \pi) \), that is, \( \int_{-\pi}^{\pi} |f(\phi)|^2 \, d\phi = \|f\|_1^2 < \infty \). The correct formulation of the Dirichlet problem in this case is

\[
\nabla^2 u = 0, \quad r < 1; \\
\lim_{r \to 1^-} \|u(r, \phi) - f(\phi)\|_1 = 0.
\]

(6.5)

The second condition in (6.5) is just

\[
\lim_{r \to 1^-} \left[ \int_{-\pi}^{\pi} |u(r, \phi) - f(\phi)|^2 \, d\phi \right]^{1/2} = 0.
\]

Solution by Separation of Variables

We shall seek the explicit solution of (6.4) or (6.5) by the method of separation of variables described below. The steps which lead to this solution need not be the subject of careful scrutiny as long as we check afterward that the alleged solution actually satisfies (6.4) or (6.5), is unique, and depends continuously on the boundary data \( f(\phi) \).

The basic building blocks in constructing this solution are the separable solutions of Laplace's equation in polar coordinates—solutions of the form \( R(r)\Phi(\phi) \). By a suitable superposition of solutions of this type we shall then obtain the required solution of (6.5) (this solution will usually not itself be separable). Substituting \( u^* = R(r)\Phi(\phi) \) in Laplace's equation, we find

\[
\frac{r}{R} \frac{d}{dr} \left( r \frac{dR}{dr} \right) = -\frac{1}{\Phi} \frac{d^2\Phi}{d\phi^2}.
\]

The left side depends on \( r \) alone and the right side on \( \phi \) alone. Since the variables \( r \) and \( \phi \) are independent, the two sides must equal the same constant,
labeled \( \lambda \) and known as a separation constant. Thus we are led to the two ordinary differential equations:

\[
- \frac{d^2 \Phi}{d \varphi^2} = \lambda \Phi, \tag{6.6}
\]

\[
r \frac{d}{dr} \left( r \frac{dR}{dr} \right) = \lambda R. \tag{6.7}
\]

Since \( u^* = R \Phi \) is harmonic in \( D \), we must demand that \( u^* \) and \( \nabla \Phi \) be continuous in \( D \), for otherwise \( \nabla^2 u^* \) would not exist. The points directly above and below the negative real axis are described by \( \varphi = \pi^- \) and \( \varphi = -\pi^+ \), respectively. To ensure continuity of \( u^* \) and \( \nabla \Phi \) on this line we must stipulate the conditions

\[
\Phi(\pi^-) = \Phi(-\pi^+),
\]

\[
\frac{d\Phi}{d\varphi}(\pi^-) = \frac{d\Phi}{d\varphi}(-\pi^+),
\]

which will serve as boundary conditions for (6.6). These so-called periodic boundary conditions apply only when dealing with a full \( 2\pi \) range for \( \varphi \) (as is the case for the interior of a circle, the exterior of a circle, or a circular annulus).

Thus we are led to the eigenvalue problem for the \( \varphi \) equation:

\[
- \Phi'' = \lambda \Phi, \quad -\pi < \varphi < \pi; \quad \Phi(-\pi) = \Phi(\pi); \quad \Phi'(-\pi) = \Phi'(\pi).
\]

The eigenvalues are \( \lambda = \lambda_n = n^2, \quad n = 0, 1, 2, \ldots \) with eigenfunctions \( \Phi_n = C \sin n\varphi + D \cos n\varphi \) (or \( A e^{in\varphi} + B e^{-in\varphi} \)) for \( n > 0 \) and \( \Phi_0 = A \). For many purposes it is more convenient to let the index \( n \) also assume negative values and to think of the single eigenfunction \( e^{in\varphi} \) as being attached to the index \( n \).

Setting \( \lambda = n^2 \) in the radial equation (6.7), we obtain the solutions

\[
R_n(r) = C r^n + D r^{-n}, \quad n \neq 0,
\]

\[
R_0(r) = C + D \log r.
\]

Since the solution must remain finite at the origin we retain only the solution \( r^n \) for \( n > 0 \), \( r^{-n} \) for \( n < 0 \), and 1 for \( n = 0 \). In all cases we can write

\[
R_n(r) = C r^{in\varphi}.
\]

This argument can, of course, be used only when the origin is included in the region where we are solving Laplace's equation.

For each integer \( n \) (positive, negative, or 0), the function

\[
u^*(r, \varphi) = r^{in\varphi} e^{in\varphi},
\]

is harmonic, continuous, and has a continuous gradient in the entire plane. The same properties also hold for the finite sum

\[
\sum_{n=-k}^{m} a_n r^{in\varphi} e^{in\varphi},
\]
where \( \{a_n\} \) are arbitrary constants. Unfortunately, to satisfy the boundary condition at \( r = 1 \), we will have to consider an infinite sum instead of a finite one, thereby incurring all the difficulties related to questions of convergence. Proceeding formally we look for a solution of the type

\[
\begin{align*}
u(r, \phi) &= \sum_{n=-\infty}^{\infty} a_n r^{|n|} e^{in\phi}, & r < 1. \quad (6.8)
\end{align*}
\]

The coefficients \( a_n \) will be determined from the boundary condition at \( r = 1 \). Before carrying out this calculation, we prove that the representation (6.8) is in fact valid for any function harmonic within the unit circle. Stated in a slightly more general form, we have

**Theorem 1.** If \( u(r, \phi) \) is harmonic in \( r < a \), we can represent \( u \) in the form

\[
\begin{align*}
u &= \sum_{n=-\infty}^{\infty} a_n r^{|n|} e^{in\phi}, & r < a.
\end{align*}
\]

**Proof.** For each fixed \( r < a \), \( u \) and \( \partial u / \partial \phi \) are continuous, so that we can expand \( u \) in the convergent Fourier series

\[
\begin{align*}
u(r, \phi) &= \sum_{n=-\infty}^{\infty} u_n(r) e^{in\phi},
\end{align*}
\]

with

\[
\begin{align*}
u_n(r) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} u(r, \phi) e^{-in\phi} \, d\phi.
\end{align*}
\]

Starting with Laplace's equation in polar coordinates, we multiply both sides by \( (1/2\pi) e^{-in\phi} \) and integrate from \( -\pi \) to \( \pi \) to obtain

\[
\begin{align*}
\frac{1}{2\pi} \int_{-\pi}^{\pi} \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \phi^2} \right] e^{-in\phi} \, d\phi &= 0, & r < a.
\end{align*}
\]

We integrate the second term by parts twice and use the fact that \( u(r, \pi) = u(r, -\pi) \) and \( \partial u / \partial \phi(r, \pi) = \partial u / \partial \phi(r, -\pi) \) to find

\[
\begin{align*}
\frac{1}{r} \left[ ru_n'(r) \right]' - \frac{n^2}{r^2} u_n &= 0.
\end{align*}
\]

For \( n \neq 0 \) and \( n = 0 \), the solutions which are finite at \( r = 0 \) are \( A r^{|n|} \) and \( A \), respectively. It therefore follows that for \( r < a \), the representation (6.8) must hold.

In a similar way one can find a representation for functions harmonic in the exterior of a circle.

**Theorem 2.** If \( u(r, \phi) \) is harmonic in \( r > a \) and \( u \) is bounded at infinity, we have

\[
\begin{align*}
u &= \sum_{n=-\infty}^{\infty} a_n r^{-|n|} e^{in\phi}, & r > a.
\end{align*}
\]
The constant term corresponding to \( n = 0 \) vanishes if \( u \) is required to vanish at infinity.

We now return to the solution of the Dirichlet problem. In some appropriate sense we want \( u(1, \varphi) = f(\varphi) \). This suggests that

\[
f(\varphi) = \sum_{n=-\infty}^{\infty} a_n e^{in\varphi},
\]

and, therefore,

\[
a_n = f_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\varphi) e^{-in\varphi} \, d\varphi.
\]

Thus we are led to the following explicit candidate for the solution of (6.4) or (6.5):

\[
u(r, \varphi) = \sum_{n=-\infty}^{\infty} f_n r |n| e^{in\varphi}
\]  \hspace{1cm} (6.9)

where

\[
f_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\varphi) e^{-in\varphi} \, d\varphi.
\]  \hspace{1cm} (6.10)

Before proceeding with the verification, we transform (6.9) to an expression that reveals more clearly the dependence of \( u(r, \varphi) \) on the boundary temperature \( f(\varphi) \). Substituting (6.10) in (6.9) [and prudently renaming the dummy variable in (6.10) so as not to confuse it with the primary variable \( \varphi \) in (6.9)], we obtain

\[
u(r, \varphi) = \int_{-\pi}^{\pi} \left[ \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} r |n| e^{in(\varphi - \psi)} \right] f(\psi) \, d\psi.
\]

The interchange of summation and integration is permissible because, for fixed \( r < 1 \), the series (6.9) clearly converges uniformly in \(-\pi < \varphi < \pi\).

Now, for any complex \( z \) (\(|z| < 1\)) and for any real \( a \),

\[
\sum_{n=-\infty}^{\infty} z^n e^{ina} = 1 + \sum_{n=1}^{\infty} z^n e^{ina} + \sum_{n=1}^{\infty} z^n e^{-ina} = 1 + \frac{ze^{ia}}{1 - ze^{ia}} + \frac{ze^{-ia}}{1 - ze^{-ia}} = \frac{1 - z^2}{1 + z^2 - 2z \cos a},
\]

so that

\[
u(r, \varphi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - r^2}{1 + r^2 - 2r \cos(\varphi - \psi)} f(\psi) d\psi, \quad r < 1.
\]  \hspace{1cm} (6.11)
Formulas (6.11) and (6.9) are alternative representations of the solution of the Dirichlet problem (6.5).

The function

\[ \frac{1}{2\pi} \frac{1 - r^2}{1 + r^2 - 2r \cos(\varphi - \psi)} \]  

(6.12)

is known as the Poisson kernel and can be interpreted as a boundary influence function. Indeed, (6.12) is the temperature at the interior point \((r, \varphi)\) caused by a boundary temperature which is a delta function concentrated at \(\psi\). The expression (6.11) is then obtained by representing \(f\) as a superposition of delta functions.

**Verification of the Solution of (6.5)**

With \(f(\varphi)\) in \(L_2(-\pi, \pi)\) we wish to show that \(u(r, \varphi)\), as given by the equivalent formulas (6.9) and (6.11), actually satisfies (6.5); that this solution is unique; and that it depends continuously on the boundary data \(f(\varphi)\).

Considered as a function of \(r\) and \(\varphi\), the Poisson kernel (6.12) is singular only at the point \(r = 1, \varphi = \psi\). Except at that point the Poisson kernel is easily seen to be harmonic. By differentiation under the integral sign, it follows that \(u(r, \varphi)\), as given by (6.11), is harmonic for \(r < 1\) and for \(r > 1\).

Next we wish to show that \(u(r, \varphi)\) satisfies the boundary condition in the form given in (6.5). We start with the series representation (6.9). First we prove that this series converges in the mean. Letting

\[ u_n(r, \varphi) = \sum_{k=-n}^{n} f_k r^{|k|} e^{ik\varphi}, \]

we find, for \(n \geq m \geq 0\),

\[ \|u_n - u_m\|_1^2 = 2\pi \sum_{k=m}^{n} |f_k|^2 |r^{2k}| + 2\pi \sum_{k=-n}^{-m} |f_k|^2 |r^{2k}| \]

\[ \leq 2\pi \sum_{k=m}^{n} |f_k|^2 + 2\pi \sum_{k=-n}^{-m} |f_k|^2, \quad r \leq 1. \]

Since \(\sum |f_k|^2\) converges, we conclude that \(u_n\) is a Cauchy sequence for each fixed \(r \leq 1\); hence \(u_n\) must converge in the mean to some element which we call \(u(r, \varphi)\). The convergence is clearly uniform in \(r\) for \(r \leq 1\). We also have

\[ \|u_n(r, \varphi) - u_n(1, \varphi)\|_1^2 = 2\pi \sum_{k=-n}^{n} |f_k|^2 |r^{1k} - 1|^2. \]

Since \(\lim_{r \to 1} (r^{1k} - 1) = 0\), it follows that, for each fixed \(n\),

\[ \lim_{r \to 1} \|u_n(r, \varphi) - u_n(1, \varphi)\|_1 = 0. \]

Now consider

\[ \|u - f\|_1 \leq \|u(r, \varphi) - u_n(r, \varphi)\|_1 + \|u_n(r, \varphi) - u_n(1, \varphi)\|_1 + \|u_n(1, \varphi) - f\|_1. \]
Given \( \varepsilon > 0 \), we can choose \( n \) large enough so that the first and third terms on the right side are smaller than \( \varepsilon/3 \); with \( n \) so chosen, the second term can be made less than \( \varepsilon/3 \) by choosing \( r \) sufficiently close to 1. Therefore, for \( r \) sufficiently near 1,

\[
\|u - f\|_1 < \varepsilon,
\]

which shows that \( u \) assumes the boundary value \( f \) in the sense of (6.5).

To prove uniqueness of the solution, let \( v \) be the difference between two solutions \( u_1 \) and \( u_2 \) of the Dirichlet problem (6.5). Then \( \nabla^2 v = 0 \) for \( r < 1 \), and

\[
\|v(r, \varphi)\|_1 \leq \|u_1 - f\|_1 + \|f - u_2\|_1.
\]

It follows that

\[
\lim_{r \to 1} \|v(r, \varphi)\|_1^2 = 0. \tag{6.13}
\]

Since \( v \) is harmonic for \( r < 1 \), it has, by Theorem 1, the representation

\[
v = \sum_{n=-\infty}^{\infty} b_n r^{|n|} e^{in\varphi}
\]

\[
\|v(r, \varphi)\|_1^2 = 2\pi \sum_{n=-\infty}^{\infty} |b_n|^2 r^{2|n|}.
\]

Using (6.13), we find \( b_n = 0 \) and hence, \( v \equiv 0 \), \( r < 1 \), and this proves uniqueness.

We turn next to the question of continuous dependence on the boundary data. Equation (6.11) or (6.9) can be viewed as defining a linear transformation from functions \( f(\varphi) \) defined on the one-dimensional interval \( -\pi < \varphi < \pi \), into functions \( u(r, \varphi) \) defined in the interior of the unit circle \( D \). The functions \( f(\varphi) \) are assumed to be in \( \mathcal{L}_2(-\pi, \pi) \), that is,

\[
\|f\|_1^2 = \int_{-\pi}^{\pi} |f|^2 \, d\varphi < \infty.
\]

The usual Lebesgue norm over \( D \) is defined as

\[
\|u\|_2^2 = \int_{D} |u|^2 \, dx = \int_{-\pi}^{\pi} \int_{0}^{1} |r|^2 |u|^2 \, dr \, d\varphi.
\]

According to (6.11) or (6.9), we can write

\[
u = Af,
\]

where we wish to show that \( A \) is a continuous transformation. In line with the arguments presented in Chapter 2, it will suffice to show that \( A \) is bounded. We must prove that there exists a constant \( C \) such that

\[
\|u\|_2 \leq C\|f\|_1, \quad \text{for all } f.
\]
From (6.9), we find
\[
\|f\|_1^2 = 2\pi \sum_{n=-\infty}^{\infty} |f_n|^2 \\
\|u\|_2^2 = 2\pi \int_0^1 r \sum_{n=-\infty}^{\infty} |f_n|^2 r^{|n|} dr \\
= 2\pi \sum_{n=-\infty}^{\infty} \frac{1}{2|n| + 2} |f_n|^2 \leq \frac{1}{2} \|f\|_1^2.
\]

We have therefore shown that the solution of (6.5) depends continuously on the data.

**Verification of the Solution of (6.4)**

With \(f\) continuous, we wish to show that \(u(r, \varphi)\), as given by the equivalent formulas (6.9) and (6.11), satisfies (6.4); that this solution is unique and that it depends continuously on the data in some appropriate sense.

We have already shown that \(\nabla^2 u = 0\) for \(r < 1\). Next we must prove that the function given by (6.11) for \(r < 1\) and equal to \(f(\varphi)\) for \(r = 1\) is continuous in \(r \leq 1\). Only the continuity at the boundary poses a problem. Let \(\varphi_0\) be fixed; we must show that for each \(\varepsilon > 0\),
\[
|u(r, \varphi) - f(\varphi_0)| < \varepsilon,
\]
whenever the point \((r, \varphi)\) is sufficiently close to the point \((1, \varphi_0)\).

We begin by rewriting (6.11) as
\[
u(r, \varphi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - r^2}{1 + r^2 - 2r \cos \theta} f(\theta + \varphi) d\theta,
\]
which results from the change of variables \(\theta = \psi - \varphi\). In this integral, the argument of \(f\) runs only between \(-\pi\) and \(\pi\) and we are at liberty to define \(f\) as we please for other values of the argument. We extend \(f\) as a function of period \(2\pi\); since \(f(-\pi) = f(\pi)\), our extended function will be continuous in \((-\infty, \infty)\). Since \(\cos \theta\) also has period \(2\pi\), we can integrate over any \(2\pi\) interval, so that
\[
u(r, \varphi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - r^2}{1 + r^2 - 2r \cos \theta} f(\theta + \varphi) d\theta = \int_{-\infty}^{\infty} s_r(\theta) f(\theta + \varphi) d\theta, \tag{6.14}
\]
where
\[
s_r(\theta) = \begin{cases} 
\frac{1}{2\pi} \frac{1 - r^2}{1 + r^2 - 2r \cos \theta}, & |\theta| < \pi; \\
0, & |\theta| \geq \pi.
\end{cases} \tag{6.15}
\]
We now show that
\[ \lim_{r \to 1^-} u(r, \varphi) = f(\varphi), \quad \text{uniformly in } -\pi < \varphi < \pi. \]

What is essentially involved here is a slight refinement of an argument which shows that \( s_r(\theta) \) is a \( \delta \) sequence, that is, \( \lim_{r \to 1^-} s_r(\theta) = \delta(\theta) \). It is worthwhile to repeat the proof as it applies in our particular case. We observe that \( s_r(\theta) \geq 0 \) for \( -\infty < \theta < \infty \) and \( 0 < r < 1 \). Moreover, Exercise 6.2 shows that
\[ \int_{-\infty}^{\infty} s_r(\theta) d\theta = 1, \quad 0 < r < 1. \]

Therefore,
\[ u(r, \varphi) - f(\varphi) = \int_{-\infty}^{\infty} s_r(\theta) \eta(\theta, \varphi) d\theta, \]
where
\[ \eta(\theta, \varphi) = f(\theta + \varphi) - f(\varphi). \]

We break up the last integral into three parts:
\[ \int_{-\infty}^{\infty} \eta_s \, d\theta = \int_{-\infty}^{-A} \eta_s \, d\theta + \int_{A}^{\infty} \eta_s \, d\theta + \int_{-A}^{A} \eta_s \, d\theta = I_1 + I_2 + I_3. \]

Since \( s_r \) is nonnegative, we can estimate \( I_3 \) as follows:
\[ |I_3| \leq \left[ \max_{-A < \theta < A} |\eta| \right] \int_{-A}^{A} s_r \, d\theta \leq \max_{-A < \theta < A} |\eta|. \]

The fact that \( f(\varphi) \) is uniformly continuous in \( -\infty < \varphi < \infty \) implies that for any \( \varepsilon > 0 \) we can choose \( A \) so small that
\[ |\eta| = |f(\theta + \varphi) - f(\varphi)| < \varepsilon, \quad \text{whenever } |\theta| < A, \]
where the choice of \( A \) can be made independent of \( \varphi \).

With \( A \) so chosen, we turn to the estimation of \( I_1 + I_2 \). Since \( f(\varphi) \) has period \( 2\pi \) and is bounded in \( -\pi < \varphi < \pi \), it must be bounded in \( -\infty < \varphi < \infty \). Therefore, \( \eta \) is bounded for all \( \theta \) and \( \varphi \), say \( |\eta| < P \), and
\[ |I_1 + I_2| \leq P \left[ \int_{-\pi}^{-A} s_r \, d\theta + \int_{A}^{\pi} s_r \, d\theta \right]. \]

The maximum of \( s_r \) in the intervals of integration occurs at \( \theta = \pm A \). This maximum goes to 0 as \( r \) approaches 1. Thus \( s_r \to 0 \) uniformly over the finite intervals \( (-\pi, -A) \) and \( (A, \pi) \). Therefore the integrals also approach 0. Hence, for any \( \varepsilon > 0 \), we can choose \( R \) near enough to 1 so that
\[ |I_1 + I_2| < \varepsilon \quad \text{for all } r \text{ such that } R < r < 1. \]

Thus for all \( r, R < r < 1 \), and all \( \varphi \),
\[ |u(r, \varphi) - f(\varphi)| < 2\varepsilon, \]

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which shows that \( u(r, \varphi) \) converges uniformly to \( f(\varphi) \) as \( r \to 1^- \).

Now
\[
|u(r, \varphi) - f(\varphi_0)| \leq |u(r, \varphi) - f(\varphi)| + |f(\varphi) - f(\varphi_0)|.
\]

For any \( \varepsilon > 0 \), we can find \( R \) such that
\[
|u(r, \varphi) - f(\varphi)| < \varepsilon \quad \text{for all } \varphi \text{ and } R < r < 1,
\]
and by the continuity of \( f \) there exists \( \delta > 0 \) such that
\[
|f(\varphi) - f(\varphi_0)| < \varepsilon, \quad \text{whenever } |\varphi - \varphi_0| < \delta.
\]

Therefore,
\[
|u(r, \varphi) - f(\varphi_0)| < 2\varepsilon, \quad \text{whenever } |\varphi - \varphi_0| < \delta \text{ and } R < r < 1.
\]

This, of course, is just what we wanted to prove.

The discussion of uniqueness and continuous dependence on the boundary values is postponed until Section 6.3.

Suppose that a function \( u(r, \varphi) \) is harmonic in \( r < 1 \); then from the differential equation (6.2) it follows that \( v(r, \varphi) = u(r/a, \varphi) \) is harmonic in \( r < a \). Moreover, if \( u(1, \varphi) = f(\varphi) \), \( v(a, \varphi) = f(\varphi) \). Therefore the solution of the interior Dirichlet problem for a circle of radius \( a \) can be expressed in either of the two forms:
\[
v(r, \varphi) = \sum_{n=-\infty}^{\infty} f_n \left( \frac{r}{a} \right)^{|n|} e^{in\varphi}
\]  \hspace{1cm} (6.16)

or
\[
v(r, \varphi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{a^2 - r^2}{a^2 + r^2 - 2ar \cos(\varphi - \psi)} f(\psi) d\psi.
\]  \hspace{1cm} (6.17)

6.3 SOME PROPERTIES OF HARMONIC FUNCTIONS

**Mean Value Theorem**

From either (6.16) or (6.17), we note the important *mean value property*
\[
v|_{r=0} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\psi) d\psi,
\]  \hspace{1cm} (6.18)

which states that the temperature at the center of a circle is the average of the boundary temperature. We also conclude that the temperature at the origin is the average of the temperature on the circumference of any circle with center at the origin and radius less than \( a \). In fact, if \( R \) is any plane region whatever, it follows that a harmonic function in \( R \) has the same average value on all concentric circles whose interiors lie wholly in \( R \).
It is worthwhile to rederive (6.18) by arguments which do not depend on a knowledge of the explicit solutions (6.16) and (6.17). Consider a circle of radius $a$ whose circumference has been divided into $n$ equal parts $C_1, \ldots, C_n$. We shall discuss $n$ separate boundary value problems, which we shall call part-problems. In the $k$th part-problem the boundary temperature is 1 on $C_k$ and 0 on the rest of the circumference. It is quite clear that any part-problem can be obtained from any other part-problem by an appropriate rotation about the center of the circle. Since the operator $\nabla^2$ is invariant under such a rotation, the temperature at the center of the circle is the same for each part-problem. On the other hand, by superposition, the sum of the temperatures for these $n$ part-problems is the temperature in a circle whose entire boundary is at temperature 1. The solution of this last problem is $u \equiv 1$ throughout the circle and, in particular, the central temperature is 1. It therefore follows that the central temperature for each part-problem is $1/n$. Now consider an arbitrary boundary temperature $f(\phi)$ and let the $n$ equal parts $C_1, \ldots, C_n$ be indexed consecutively in counterclockwise order starting from $\phi = -\pi$. Thus $C_k$ is subtended by an angle $\Delta \phi = 2\pi/n$ and the $\phi$ interval which corresponds to $C_k$ is

$$\varphi_{k-1} < \phi < \varphi_k,$$

where

$$\varphi_k = -\pi + \frac{2\pi k}{n}.$$

If $E_k(\phi)$ stands for the function which is equal to 1 for $\varphi_{k-1} < \phi < \varphi_k$ and vanishes for other $\phi$, we have

$$f(\phi) = \sum_{k=1}^{n} f(\phi)E_k(\phi).$$

For $n$ large, the arc $C_k$ is small, and if $f(\phi)$ is continuous we may replace $f(\phi)E_k(\phi)$ by $f(\phi_k)E_k(\phi)$. The central temperature corresponding to the boundary temperature $f(\phi_k)E_k(\phi)$ is $f(\phi_k)/n$, so that, by superposition, we find that the central temperature corresponding to the boundary temperature

$$\sum_{k=1}^{n} f(\phi_k)E_k(\phi)$$

is

$$\sum_{k=1}^{n} \frac{f(\phi_k)}{n} = \sum_{k=1}^{n} \frac{1}{2\pi} f(\phi_k)\Delta \phi.$$

Thus, as $n \to \infty$, we find that the central temperature corresponding to the boundary temperature $f(\phi)$ is

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(\phi) d\phi,$$
which confirms the result (6.18). The present derivation can be extended, with slight modifications, to any number of dimensions and to other partial differential equations which are spherically symmetric—that is, which are invariant under rotation (see Exercises 6.5 and 6.6, for instance).

In particular, we find that for Laplace’s equation in \( n \) dimensions the mean value theorem is still valid: The value of a harmonic function at the center of a sphere is the average of its values on the surface of the sphere.

**The Maximum Principle for Harmonic Functions**

In what follows \( R \) is a bounded, open region; \( \sigma \) is its boundary and \( \overline{R} \) is the closed region which is the union of \( R \) and \( \sigma \). A deep result concerning solutions of Laplace’s equation is contained in the maximum principle:

**Theorem.** If \( u \) is harmonic in \( R \) and continuous in \( \overline{R} \), the maximum and minimum values of \( u \) in \( \overline{R} \) are both attained on \( \sigma \). Moreover, if \( u \) is not constant in \( R \), the extremal values are attained only on \( \sigma \).

**Proof.** We shall prove the theorem for a plane region, but the same method is easily extended to higher dimensions.

Let us denote the average of a function \( f \) over a curve \( C \) by \( \hat{f}_C \). It is obvious that if \( f \leq g \) on \( C \), then \( \hat{f}_C \leq \hat{g}_C \); further, if \( f \) and \( g \) are continuous on \( C \) and \( f < g \) at any point on \( C \), then \( \hat{f}_C < \hat{g}_C \).

Now suppose that the maximum of the harmonic function \( u \) is attained at the interior point \( x_0 \) (as in Figure 6.2). Then \( u(x) \leq u(x_0) \) for all \( x \) in \( \overline{R} \).

![Figure 6.2](image)

Consider any circle lying wholly in \( \overline{R} \) and with center at \( x_0 \). If at some point on the circumference \( C \) of this circle we had \( u(x) < u(x_0) \), then \( \hat{u}_C < u(x_0) \), which would violate the mean value theorem. Thus \( u \) must have the constant value \( u(x_0) \) on \( C \) and therefore must equal \( u(x_0) \) within the largest circle \( C_0 \) with center at \( x_0 \) and lying entirely in \( \overline{R} \). We now show that \( u \) must have this
same value $u(x_0)$ everywhere else in $R$ (and hence, by continuity, in $\overline{R}$). Consider a point $\xi$ in $R$ and connect $\xi$ to $x_0$ by a curve lying in $R$, as shown in Figure 6.2. The intersection of this curve and $C_0$ is denoted by $x_1$; next, construct the largest circle, with center at $x_1$, lying in $\overline{R}$. By the argument previously used we find again that $u = u(x_1) = u(x_0)$ within this circle. Continuing in a similar manner we finally cover $\xi$ by some circle and therefore $u(\xi) = u(x_0)$. Hence $u$ is constant in $R$ and also in $\overline{R}$. We conclude that if $u$ is harmonic in $R$ and continuous in $\overline{R}$ and not identically constant, its maximum occurs on the boundary and not in the interior. The same argument applied to the harmonic function $-u$ shows that the minimum of $u$ also occurs on the boundary.

**Uniqueness and Continuous Dependence on Data for the Dirichlet Problem**

Let $R$ be a bounded open region in $n$ dimensions and consider the problem

$$\nabla^2 \varphi = 0 \text{ in } R; \quad \varphi \text{ continuous in } \overline{R}; \quad \varphi|_\sigma = 0.$$  \hspace{1cm} (6.19)

By the maximum principle, $\varphi \equiv 0$ in $\overline{R}$, and system (6.19) has only the trivial solution.

We now give another proof of this result not based on the maximum principle. Multiply the differential equation in (6.19) by $\varphi$ and integrate over $R$ to obtain

$$0 = \int_R (\varphi \nabla^2 \varphi) dx = -\int_R |\text{grad } \varphi|^2 dx + \int_\sigma \varphi \frac{\partial \varphi}{\partial n} dS.$$ 

Since $\varphi$ vanishes on $\sigma$, we find

$$\int_R |\text{grad } \varphi|^2 dx = 0.$$ 

The integrand is continuous and nonnegative so that

$$\text{grad } \varphi \equiv 0, \quad \text{in } R;$$

$$\varphi = C, \quad \text{in } R.$$ 

Now $\varphi|_\sigma = 0$ and $\varphi$ is continuous in $\overline{R}$, so that $C = 0$ and $\varphi \equiv 0$ in $\overline{R}$.

The disadvantage of this method is that it requires some information on the derivatives of $\varphi$ not available a priori. In fact, we have set $\int_\sigma \varphi(\partial \varphi/\partial n) dS = 0$ and this follows only if $\partial \varphi/\partial n$ has finite values on the boundary. It can be shown that if we add to (6.19) the condition that the first derivatives of $\varphi$ are continuous in $\overline{R}$, our proof becomes valid.

Consider next the Dirichlet problem in the form analogous to (6.4), with $f$ a given continuous function on $\sigma$:

$$\nabla^2 u = 0 \text{ in } R, \quad u \text{ continuous in } \overline{R}; \quad u|_\sigma = f.$$  \hspace{1cm} (6.20)
Theorem (Uniqueness of Solution). The system (6.20) has at most one solution.

Proof. If (6.20) had two solutions $u_1$ and $u_2$, their difference would satisfy (6.19), which has only the trivial solution.

The proof just given applies equally well to the Dirichlet problem for the Poisson equation $-\nabla^2 u = q(x)$. The proof of existence of a solution to the Dirichlet problem for either the Laplace or the Poisson equation is much more difficult. Some partial answers will be given in Section 6.5.

Turning next to the question of continuous dependence on the boundary data for (6.20), we introduce the following norms:

(a) For continuous functions $u$ defined on $\bar{R}$,
$$\|u\|_2 = \max_{x \in \bar{R}} |u|.$$

(b) For continuous functions $f$ defined on $\sigma$,
$$\|f\|_1 = \max_{x \in \sigma} |f|.$$

Definition. Let $u_1$ and $u_2$ be the solutions of (6.20) corresponding to $f_1$ and $f_2$, respectively. We have continuous dependence on the boundary data if to each $\varepsilon > 0$ there exists $\delta > 0$ such that
$$\|u_2 - u_1\|_2 \leq \varepsilon, \quad \text{whenever } \|f_2 - f_1\|_1 < \delta.$$

Now $u = u_2 - u_1$ satisfies (6.20) with $f = f_2 - f_1$. By the maximum principle it follows that
$$\|u\|_2 \leq \|f\|_1.$$

Hence whenever $\|f\|_1 < \varepsilon$, we have $\|u\|_2 < \varepsilon$, and therefore continuous dependence on the data has been established for (6.20).

Exercises

6.1 For any complex number $z$ with $|z| < 1$, we have
$$\sum_{n=0}^{\infty} z^n = \frac{1}{1 - z}.$$

In what follows, $\alpha$ is a complex number with $|\alpha| < 1$, and $\theta$ is an arbitrary real number. Show that

(a) $\sum_{n=1}^{\infty} \alpha^n \cos n\theta = \frac{\alpha \cos \theta - \alpha^2}{1 + \alpha^2 - 2\alpha \cos \theta}$; $\sum_{n=1}^{\infty} \alpha^n \sin n\theta = \frac{\alpha \sin \theta}{1 + \alpha^2 - 2\alpha \cos \theta}$.

(b) $\sum_{n=1}^{\infty} \frac{z^n}{n} = -\log(1 - z)$. 

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6.2 By setting $e^{i\psi} = z$, transform the real integral

$$J = \int_{-\pi}^{\pi} \frac{1 - r^2}{1 + r^2 - 2r \cos \psi} \, d\psi$$

to a counterclockwise integral on the boundary of the unit circle in the complex $z$ plane and obtain

$$J = \frac{i(1 - r^2)}{r} \int \frac{dz}{(z - r)(z - 1/r)}.$$

If $0 < r < 1$, there is only the simple pole at $z = r$ in the interior of the circle, so that

$$J = 2\pi.$$

For $r > 1$, show that

$$J = -2\pi.$$

6.3 (a) Show that if $u(r, \varphi)$ is harmonic for $r < 1$, then $u(1/r, \varphi)$ is harmonic for $r > 1$ (here $r, \varphi$ are the usual polar coordinates).

(b) Substituting $1/r$ for $r$ in (6.11), we obtain a harmonic function in $r > 1$,

$$v(r, \varphi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{r^2 - 1}{1 + r^2 - 2r \cos(\varphi - \psi)} f(\psi) \, d\psi. \quad (6.23)$$

To find $\lim_{r \to 1^+} v(r, \varphi)$, let $r = 1/\tau$. Then

$$\lim_{r \to 1^+} v(r, \varphi) = \lim_{\tau \to 1^-} v\left(\frac{1}{\tau}, \varphi\right) = \lim_{\tau \to 1^-} u(\tau, \varphi) = f(\varphi).$$

Thus (6.23) provides a solution of the exterior Dirichlet problem

$$\nabla^2 v = 0, \quad r > 1; \quad v(1, \varphi) = f(\varphi).$$

We should observe that $v(r, \varphi)$ approaches the finite limit

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(\psi) \, d\psi \quad \text{as} \quad r \to \infty.$$
symmetry arguments similar to those used in Section 6.3 to prove the mean value theorem, show that the temperature at the center of the polygon is \( k/n \).

6.5 Let \( k \) be a positive number and consider the Dirichlet problem for the Helmholtz equation in the interior of a circle of radius \( a \):

\[
\nabla^2 u + k^2 u = 0, \quad r < a; \quad u(a, \varphi) = f(\varphi).
\]

(a) Show that the solution corresponding to the constant boundary value 1 is

\[
\frac{J_0(kr)}{J_0(ka)}.
\]

By using symmetry arguments similar to those of Section 6.3, derive the mean value theorem for the Helmholtz equation

\[
u(r = 0) = \frac{1}{2\pi J_0(ka)} \int_{-\pi}^{\pi} f(\varphi) d\varphi.
\]  \hspace{1cm} (6.24)

(b) By separation of variables find the solution \( u \) for \( r < a \) as the series

\[
\sum_{n=-\infty}^{\infty} \frac{J_n(kr)}{J_n(ka)} f_n e^{in\varphi},
\]

and verify (6.24).

6.6 Consider the following problem for the biharmonic equation in the interior of a circle of radius \( a \):

\[
\nabla^2 \nabla^2 u = 0, \quad r < a; \quad u(a, \varphi) = f(\varphi); \quad \frac{\partial u}{\partial r}(a, \varphi) = g(\varphi).
\]

(a) Show that the solutions corresponding to the choices \( f = 1, g = 0 \)
and \( f = 0, g = 1 \) are, respectively,

\[
u = 1 \quad \text{and} \quad u = \frac{r^2 - a^2}{2a}.
\]

(b) By the symmetry arguments of Section 6.3, derive the mean value theorem for (I):

\[
u(r = 0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\varphi) d\varphi - \frac{a}{4\pi} \int_{-\pi}^{\pi} g(\varphi) d\varphi.
\]

6.7 Let \( P(x) \) be a positive function in the bounded open region \( R \). Consider the Dirichlet problem

\[
\nabla^2 u - Pu = 0 \text{ in } R, \quad u \text{ continuous in } \bar{R}; \quad u|_{\partial} = f.
\]
Show that if \( f \) is positive anywhere on \( \sigma \), \( u \) must attain its maximum on \( \sigma \); if \( f \) is negative anywhere on \( \sigma \), \( u \) must attain its minimum on \( \sigma \). Use this result to prove that the Dirichlet problem has at most one solution.

6.8 Let \( k(x) \) be a nonnegative function of position in the bounded open region \( R \). Prove uniqueness of the Dirichlet problem:

\[
\text{div}(k \text{ grad } u) = 0 \text{ in } R, \quad u \text{ continuous in } \overline{R}; \quad u|_\sigma = f.
\]

6.9 (a) Let \( s_r(\varphi) \) be the Poisson kernel (6.15); that is,

\[
s_r(\varphi) = \frac{1}{2\pi} \frac{1 - r^2}{1 + r^2 - 2r \cos \varphi}, \quad -\pi < \varphi < \pi.
\]

It is clear that \( s_r(\varphi) \) is harmonic in \( r < 1 \); moreover, \( \lim_{r \to 1^-} s_r(\varphi) = 0 \) for every \( \varphi \) except \( \varphi = 0 \). In the sense of distributions, we have

\[
\lim_{r \to 1^-} s_r(\varphi) = \delta(\varphi), \quad -\pi < \varphi < \pi.
\]

This last equation means, that for every test function \( a(\varphi) \),

\[
\lim_{r \to 1^-} \int_{-\pi}^{\pi} s_r(\varphi)a(\varphi)d\varphi = a(0),
\]

a statement which was proved (in a stronger form) following (6.15). Thus we can think of \( s_r(\varphi) \) as the solution of the “distributional” Dirichlet problem

\[
\nabla^2 u = 0, \quad r < 1; \quad \lim_{r \to 1^-} u(r, \varphi) = \delta(\varphi).
\]

(b) Consider the \( \varphi \) derivative of the Poisson kernel (6.15),

\[
t_r(\varphi) = \frac{\partial s_r(\varphi)}{\partial \varphi} = -\frac{1}{\pi} \frac{r(1 - r^2) \sin \varphi}{[1 + r^2 - 2r \cos \varphi]^2}, \quad -\pi < \varphi < \pi.
\]

One sees directly from (6.2) that the \( \varphi \) derivative of a harmonic function is itself harmonic. Thus \( t_r(\varphi) \) is harmonic for \( r < 1 \). Moreover, in the sense of pointwise convergence

\[
\lim_{r \to 1^-} t_r(\varphi) = 0 \quad \text{for every } \varphi.
\]

Thus \( t_r(\varphi) \) approaches 0 along every radial line! Yet \( t_r(\varphi) \) is not identically zero. This is not in violation of the uniqueness theorem for the Dirichlet problem because \( t_r(\varphi) \) is not continuous at \( r = 1, \varphi = 0 \). Indeed, if we let \( r \to 1 \) along the upper half of the semicircle \( r = \cos \varphi \), we find \( t_r \to +\infty \), whereas on the lower half \( t_r \to -\infty \). We therefore surmise that \( t_r(\varphi) \) has a dipole behavior at \( r = 1, \varphi = 0 \). This is confirmed by a distributional interpretation. In fact, we have

\[
\int_{-\pi}^{\pi} t_r a d\varphi = \int_{-\pi}^{\pi} a \frac{\partial s_r}{\partial \varphi} d\varphi = -\int_{-\pi}^{\pi} s_r \frac{da}{d\varphi} + as_r |_{-\pi}^{\pi},
\]
and, since \( \lim_{r \to 1^-} s_r(\pm \pi) = 0 \),

\[
\lim_{r \to 1^-} \int_{-\pi}^{\pi} at_r \, d\varphi = -a'(0).
\]

Thus in the distributional sense

\[
\lim_{r \to 1^-} t_r(\varphi) = \delta'(\varphi), \quad -\pi < \varphi < \pi.
\]

6.10 Consider the wedge \( R: 0 < r < 1, 0 < \varphi < \alpha \). Solve the following Dirichlet problem for \( R \):

\[
\nabla^2 u = 0 \text{ in } R; \quad u(r, 0) = u(r, \alpha) = 0, \quad 0 < r < 1;
\]

\[
u(1, \varphi) = f(\varphi), \quad 0 < \varphi < \alpha.
\]

Obtain the solution in the form

\[
u = \sum_{n=1}^{\infty} f_n \sin \frac{n\pi\varphi}{\alpha} r^{n\pi/\alpha},
\]

where

\[
f_n = \frac{2}{\alpha} \int_{0}^{\pi} f(\varphi) \sin \frac{n\pi\varphi}{\alpha} \, d\varphi.
\]

Consider the special case \( \alpha = 2\pi \). We then have a slit in a disk, the sides of the slit being at temperature 0. If \( f(\varphi) = \sin(\varphi/2) \), we find

\[
u(r, \varphi) = r^{1/2} \sin \frac{\varphi}{2}.
\]

It is worth observing that as we approach the origin (which is one end of the slit) along the radial line \( \varphi = \pi \), the temperature approaches 0 like \( r^{1/2} \). Note that the gradient of the temperature is not defined at the origin.

6.11 Solve the Dirichlet problem for the circular annulus \( a < r < b \).

6.12 Consider the curved quadrilateral \( R: a < r < b, 0 < \varphi < \alpha \).

(a) Solve the Dirichlet problem

\[
\nabla^2 u = 0 \text{ in } R; \quad u(r, 0) = u(r, \alpha) = 0, \quad a < r < b;
\]

\[
u(a, \varphi) = 0, \quad u(b, \varphi) = f(\varphi), \quad 0 < \varphi < \alpha.
\]

(b) Solve the Dirichlet problem

\[
\nabla^2 u = 0 \text{ in } R; \quad u(r, 0) = 0; \quad u(r, \alpha) = f(r), \quad a < r < b;
\]

\[
u(a, \varphi) = u(b, \varphi) = 0, \quad 0 < \varphi < \alpha.
\]
Do part (b) in two ways; first by expanding in appropriate functions of \( r \), then by expanding in the functions \( \sin n\pi \varphi /\alpha \). Care must be exercised in this latter method to handle the inhomogeneous boundary condition at \( \varphi = \alpha \). [Hint: Multiply \( \nabla^2 u = 0 \) by \( \sin(n\pi \varphi)/\alpha \), and integrate by parts, paying attention to the integrated terms.]

6.13 Consider the rectangle \( R: 0 < x_1 < a, 0 < x_2 < b \).
Show that the solution of the Dirichlet problem

\[
\nabla^2 u = 0 \text{ in } R; \quad u(0, x_2) = u(a, x_2) = 0, \quad 0 < x_2 < b;
\]

\[
u(x_1, 0) = 0; \quad u(x_1, b) = f(x_1), \quad 0 < x_1 < a
\]
is given by

\[
u(x_1, x_2) = \sum_{n=1}^{\infty} \sin \left( \frac{n\pi x_1}{a} \right) \sinh \left( \frac{n\pi x_2}{a} \right) \left[ \sinh \left( \frac{n\pi b}{a} \right) \right]^{-1} f_n,
\]

where

\[
f_n = \frac{2}{a} \int_0^a f(x_1) \sin \left( \frac{n\pi x_1}{a} \right) dx_1.
\]

Using the norm \( \| u(x_1, x_2) \|_1^2 = \int_0^a |u(x_1, x_2)|^2 dx_1 \), prove that the boundary values are assumed in the sense

\[
\lim_{x_2 \to b^-} \| u(x_1, x_2) - f(x_1) \|_1 = 0.
\]

Of course, we have assumed that \( f \) is in \( \mathcal{L}_2(0, b) \).

6.14 Let \( \varphi_n(x) \) be a complete orthonormal set in \( a < x < b \) and let \( v_n(y) \) be a set of functions with the properties

(a) \( \lim_{y \to 1^-} v_n(y) = 1 \).

(b) For all \( n \) and all \( y \) in \( y_0 < y < 1 \), \( |v_n(y)| < M \); that is, the set \( v_n(y) \) is uniformly bounded in a neighborhood of \( y = 1 \). Show that

\[
\lim_{y \to 1^-} \int_a^b \left| \sum_{k=1}^{\infty} f_k v_k(y) \varphi_k(x) - f(x) \right|^2 dx = 0,
\]

where \( f(x) \) is any function in \( \mathcal{L}_2(a, b) \) and

\[
f_k = \int_a^b f(x) \overline{\varphi_k(x)} dx.
\]

The proof is an easy generalization of the one used in verifying the solution of the Dirichlet problem for the unit circle [see after (6.12)].

6.15 Consider the Dirichlet problem for the interior of the unit sphere in three dimensions, that is,

\[
\nabla^2 u = 0, \quad r < 1; \quad u(1, \theta, \varphi) = f(\theta, \varphi),
\]
where $f(\theta, \varphi)$ is a given continuous function of the spherical coordinates $\theta$ and $\varphi$.

Using separation of variables in spherical coordinates, show that the solution can be expressed as a series of spherical harmonics (see Appendix A)

$$u(r, \theta, \varphi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{mn} r^n Y_n^m(\theta, \varphi), \quad (6.25)$$

where

$$a_{mn} = \frac{1}{N_{mn}} \int_{0}^{2\pi} d\varphi \int_{0}^{\pi} d\theta \sin \theta f(\theta, \varphi) \bar{Y}_n^m(\theta, \varphi).$$

Now let us calculate $u$ on the positive polar axis (that is, for $\theta = 0$). Our result must then be independent of $\varphi$, since $\varphi$ is indeterminate on the polar axis. From the fact that

$$P_n^{|m|}(1) = 0, \quad m \neq 0; \quad P_n^{|m|}(1) = 1, \quad m = 0,$$

we find

$$u(r, 0, \varphi) = \sum_{n=0}^{\infty} \frac{(2n+1)r^n}{4\pi} \int_{0}^{2\pi} d\varphi' \int_{0}^{\pi} d\theta' \sin \theta' f(\theta', \varphi') P_n(\cos \theta')$$

which is indeed independent of $\varphi$. By using the relation (A.14), Appendix A, obtain

$$u(r, 0, \varphi) = \frac{1 - r^2}{4\pi} \int_{0}^{2\pi} d\varphi' \int_{0}^{\pi} d\theta' \sin \theta' \frac{f(\theta', \varphi')}{(1 + r^2 - 2r \cos \theta')^{3/2}}.$$

Now any direction emanating from the origin may be chosen as the polar axis. Choose the polar axis through $\theta, \varphi$. Then if $\gamma$ is the angle between this new polar axis and the radial line through $\theta', \varphi'$, we have

$$u(r, \theta, \varphi) = \frac{1 - r^2}{4\pi} \int_{0}^{2\pi} d\varphi' \int_{0}^{\pi} d\theta' \sin \theta' \frac{f(\theta', \varphi')}{(1 + r^2 - 2r \cos \gamma)^{3/2}}.$$

It is easy to express $\cos \gamma$ in terms of $\theta, \varphi, \theta'$, and $\varphi'$. In fact, $\gamma$ is the angle between the radial lines through $(1, \theta, \varphi)$ and $(1, \theta', \varphi')$. Introducing Cartesian coordinates $x_1, x_2, x_3$ and $x'_1, x'_2, x'_3$ for these two points, we find that the distance $p$ between them is given by

$$p^2 = (x_1 - x'_1)^2 + (x_2 - x'_2)^2 + (x_3 - x'_3)^2,$$

but

$$x_1 = \sin \theta \cos \varphi, \quad x_2 = \sin \theta \sin \varphi, \quad x_3 = \cos \theta,$$

$$x'_1 = \sin \theta' \cos \varphi', \quad x'_2 = \sin \theta' \sin \varphi', \quad x'_3 = \cos \theta',$$

so that

$$p^2 = 2 - 2 \cos \theta \cos \theta' - 2 \sin \theta \sin \theta' \cos (\varphi - \varphi').$$
We also have
\[ p^2 = 2 - 2 \cos \gamma, \]
and, therefore,
\[ \cos \gamma = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\varphi - \varphi'), \]
which enables us to express the solution \( u \) in the form
\[ u(r, \theta, \varphi) = \frac{1 - r^2}{4\pi} \int_0^{2\pi} \int_0^\pi d\varphi' \int_0^\pi d\theta' \sin \theta' \]
\[ \times \frac{f(\theta', \varphi')}{\{1 + r^2 - 2r[\cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\varphi - \varphi')]\}^{3/2}}. \]

(6.26)

Formula (6.26) should be compared with (6.11).

6.16 Solve the Dirichlet problem for the exterior of the unit sphere in three dimensions when the potential at infinity is required to vanish. Obtain the expansion
\[ u(r, \theta, \varphi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} b_{mn} r^{-n-1} Y_n^m(\theta, \varphi), \]
where
\[ b_{mn} = \frac{1}{N_{mn}} \int_0^{2\pi} \int_0^\pi d\varphi \int_0^\pi d\theta \sin \theta f(\theta, \varphi) Y_n^m(\theta, \varphi). \]

6.17 Solve the Dirichlet problem for the spherical annulus \( a < r < b \). The boundary conditions are
\[ u(a, \theta, \varphi) = f_1(\theta, \varphi), \quad u(b, \theta, \varphi) = f_2(\theta, \varphi). \]

6.4 SURFACE LAYERS

In this section it will be convenient to use the language of electrostatics. The results, being purely mathematical in nature, can be applied in all the other physical contexts of potential theory. Our goal is to investigate the behavior of the potentials arising from certain elementary configurations of charge. The program will be carried out in three dimensions but there is no difficulty in adapting the methods to spaces of other dimensions.

If a unit positive charge is located at \( \xi \) in free space, the corresponding potential \( E(x | \xi) \) satisfies the differential equation
\[ -\nabla^2 E = \delta(x - \xi). \]
The solution is determined only up to an arbitrary solution of the homogeneous equation. If we require that \( E \) be spherically symmetric and vanish
at $|x| = \infty$, we find, as in Chapter 5, the unique solution

$$E(x | \xi) = \frac{1}{4\pi|x - \xi|}.$$ 

Now suppose that charge is distributed in free space with the volume density $q(x)$, where we assume that $q$ vanishes identically for $|x|$ sufficiently large. The element of volume $d\xi$ at $\xi$ carries a charge $q(\xi)d\xi$ which may be regarded as a point charge and therefore creates the potential $\frac{1}{4\pi|x - \xi|} q(\xi)d\xi$.

By superposition the potential $u(x)$ due to the entire charge distribution is the sum of the potentials arising from the contributions of individual elements, and therefore

$$u(x) = \int_{R_3} \frac{1}{4\pi|x - \xi|} q(\xi)d\xi,$$  \hspace{1cm} (6.27)

where $R_3$ is the whole of three-dimensional space.

By differentiating under the integral sign, one can show formally that

$$-\nabla^2 u = q(x).$$  \hspace{1cm} (6.28)

The distributional interpretation of (6.27) and (6.28) is completely rigorous. In fact, (6.27) is just the convolution $q \ast 1/4\pi|x|$, and by the arguments of Chapter 5 it follows that

$$-\nabla^2 u = -q \ast \nabla^2 \frac{1}{4\pi|x|} = q \ast \delta = q(x).$$

Next we want to investigate the potential of a dipole. Let $h$ be a small positive number and let charges $-1/h$ and $1/h$ be located at $\xi$ and $\xi + hl$, respectively, where $l$ is a given unit vector. The corresponding potential $u$ satisfies the equation

$$-\nabla^2 u = \frac{1}{h} \{\delta[x - (\xi + hl)] - \delta(x - \xi)\}.$$  

As $h \to 0$, the charges become infinite while the distance separating them approaches 0 in such a way that the product of the charge and distance remains equal to 1. The limiting charge configuration is called a unit dipole with axis $l$. Thus the axis of the dipole points from the negative to the positive charge. The right side of the differential equation for $u$ approaches

$$\frac{d}{dl} \delta(x - \xi),$$

where $d/dl$ refers to the directional derivative in the $l$ direction, the differentiation being carried out with respect to the variable $\xi$. By the principle of
superposition we find the dipole potential \( D_l(x | \xi) \), that is, the potential at \( x \) due to a unit dipole with axis \( l \) located at \( \xi \):

\[
D_l(x | \xi) = \lim_{l \to 0} u = \frac{d}{dl} \frac{1}{4\pi|x - \xi|} = \frac{\cos (x - \xi, l)}{4\pi|x - \xi|^2} = \frac{\cos \theta}{4\pi|x - \xi|^2}, \tag{6.29}
\]

where \( \theta \) is the angle between the vectors \( l \) and \( x - \xi \).

Suppose now that charge is distributed on the surface \( \sigma \) with surface charge density \( a(x) \). Such a charge configuration will be known as a simple layer. The element of surface \( dS_{\xi} \) at the point \( \xi \) on \( \sigma \) carries the concentrated charge \( a(\xi)dS_{\xi} \), whose potential is \( a(\xi)dS_{\xi}/4\pi|x - \xi| \). By superposition the total potential \( u \) due to the simple layer is given by

\[
u(x) = \int_{\sigma} \frac{1}{4\pi|x - \xi|} a(\xi)dS_{\xi}. \tag{6.30}\]

It is clear that at any point \( x \) not on \( \sigma \), \( u \) is well defined and \( \nabla^2 u = 0 \). Now let \( s \) be a fixed point on \( \sigma \) and consider \( u(s) \) as given by (6.30). The integral is then improper but we shall prove that it is still convergent. It will be sufficient to show that the contribution to (6.30) arising from the part of the surface near \( s \) is vanishingly small. Let \( \sigma_e \) be the part of \( \sigma \) enclosed within a small sphere of radius \( \varepsilon \) with center at \( s \). We wish to show

\[
\lim_{\varepsilon \to 0} \int_{\sigma_e} \frac{1}{4\pi|s - \xi|} a(\xi)dS_{\xi} = 0.
\]

To this end we draw the tangent plane to \( \sigma \) at \( s \) and denote by \( \xi' \), \( \sigma_e' \), \( dS' \) the projections on the tangent plane of the point \( \xi \), the surface \( \sigma_e \), and the surface element \( dS_{\xi} \), respectively. Then

\[
\int_{\sigma_e} \frac{1}{4\pi|s - \xi|} a(\xi)dS_{\xi} = \int_{\sigma_e'} \frac{1}{4\pi|s - \xi|} a(\xi) \sec \gamma dS',
\]

where \( \gamma \) is the angle between the normals to \( \sigma \) at \( s \) and \( \xi \). Since there is a one-to-one correspondence between points on \( \sigma_e' \) and \( \sigma_e \), we can write

\[
a(\xi) = a[\xi(\xi')] = a'(\xi').
\]

Moreover, if \( \varepsilon \) is small, the angle between the normals at \( s \) and \( \xi \) is small and we can therefore surely choose \( \varepsilon \) small enough so that

\[
|\sec \gamma| < 2.
\]

Further, it is clear that \( |s - \xi| \geq |s - \xi'| \) and

\[
\int_{\sigma_e'} \frac{1}{4\pi|s - \xi|} a(\xi) \sec \gamma dS' \leq 2 \int_{\sigma_e} \frac{|a'(\xi')|}{4\pi|s - \xi'|} dS'.
\]
We now introduce polar coordinates \( \rho' \) and \( \phi' \) on \( \sigma' \), with origin at \( s \). Then, since \( |s - \xi'| = \rho' \), we have

\[
\int_{\sigma'} \frac{|a'|}{4\pi|s - \xi'|} dS' \leq \int_0^{2\pi} d\phi' \int_0^\rho' d\rho' \frac{|a'|}{4\pi\rho'}.
\]

The double integral on the right side clearly approaches 0 as \( \varepsilon \to 0 \), so that (6.30) converges even when \( x \) is on \( \sigma \).

We consider next a surface distribution of dipoles normal to the surface \( \sigma \). Such a charge configuration is known as a double layer. If the surface density is \( b(x) \), the element of surface \( dS_\xi \) at \( \xi \) carries a dipole of strength \( b(\xi)dS_\xi \) with axis \( n \), where \( n \) is the normal to \( \sigma \) at \( \xi \). From (6.29), the potential of this element is

\[
\frac{\cos (x - \xi, n)}{4\pi|x - \xi|^2} b(\xi) dS_\xi.
\]

The total potential \( v(x) \) of the double layer is therefore

\[
v(x) = \int_\sigma b(\xi) \frac{\cos (x - \xi, n)}{4\pi|x - \xi|^2} dS_\xi. \tag{6.31}
\]

Whenever \( x \) is not on \( \sigma \), it is evident from (6.31) that \( v(x) \) is well defined and that \( \nabla^2 v = 0 \). Moreover, we can show that (6.31) remains a convergent integral even when \( x \) is a point \( s \) on \( \sigma \). The difficulty stems from the points \( \xi \) on \( \sigma \) near \( s \), and for such points \( \theta \) is nearly \( \pi/2 \), since \( s - \xi \) is almost perpendicular to \( n \). By performing a Taylor expansion one can show that, for \( \xi \) near \( s \), \( \cos \theta \sim C|s - \xi| \). Thus the integrand in (6.31) has a singularity at \( s = \xi \) which is of the order \( 1/|s - \xi| \). We can then use the same argument employed for (6.30) to show that (6.31) converges when \( s \) is on \( \sigma \). The situation in two dimensions is even more favorable (see Exercise 6.18).

We now want to investigate the behavior of the potential (and of derivatives of the potential) of simple and double layers as the observation point \( x \) approaches a point \( s \) on \( \sigma \). The fact that (6.30) and (6.31) retain a meaning when \( x = s \) does not help us in finding limiting values of \( u \) or \( v \) as \( x \) approaches \( s \), for it is possible that \( u \) or \( v \) might be discontinuous on \( \sigma \). We shall be able to derive the desired results by attacking the problem from a distributional point of view. We begin by a study of the fundamental solution

\[
E(x_1, x_2, x_3) = \frac{1}{4\pi r} = \frac{1}{4\pi[x_1^2 + x_2^2 + x_3^2]^{1/2}},
\]

where we have taken \( \xi = 0 \), and \( x_1, x_2, \) and \( x_3 \) are Cartesian coordinates. We are interested in the distributional behavior of \( E \) as \( x_3 \to 0 \). For this purpose we regard \( E \) as a function of the two variables \( x_1 \) and \( x_2 \), depending parametrically on \( x_3 \). For each \( x_3 \neq 0 \), \( E \) is a locally integrable function of \( x_1 \) and \( x_2 \) and therefore defines a regular distribution. For \( x_3 = 0 \), \( E \) has a singularity.
at \( x_1 = x_2 = 0 \), but \( E \) is still locally integrable, as can be seen by introducing polar coordinates. Thus for every value of \( x_3 \), \( E \) defines a regular distribution on the space \( K_2 \) of test functions \( \varphi(x_1, x_2) \) through the formula

\[
\langle E, \varphi(x_1, x_2) \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{4\pi r} \varphi(x_1, x_2) dx_1 \, dx_2.
\]

The action of \( E \) on \( \varphi \) depends on the parameter \( x_3 \). We wish to investigate

\[
\lim_{x_3 \to 0^\pm} \langle E, \varphi \rangle.
\]

First, we observe that in the sense of pointwise convergence, we have

\[
\lim_{x_3 \to 0^\pm} E(x_1, x_2, x_3) = E(x_1, x_2, 0),
\]

for all \( x_1 \) and \( x_2 \) except \( x_1 = x_2 = 0 \). Moreover,

\[
|E(x_1, x_2, x_3)| \leq E(x_1, x_2, 0)
\]

and \( E(x_1, x_2, 0) \) is locally integrable.

It therefore follows from the Lebesgue convergence theorem (Section 2.3) that

\[
\lim_{x_3 \to 0^\pm} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E(x_1, x_2, x_3) \varphi(x_1, x_2) dx_1 \, dx_2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E(x_1, x_2, 0) \varphi(x_1, x_2) dx_1 \, dx_2,
\]

(6.32)

for every function \( \varphi \) which is bounded, continuous at the origin, and vanishes outside a finite region. Thus (6.32) certainly holds for any test function \( \varphi(x_1, x_2) \), so that we can write in the distributional sense

\[
\lim_{x_3 \to 0^\pm} \frac{1}{4\pi[x_1^2 + x_2^2 + x_3^2]^{1/2}} = \frac{1}{4\pi[x_1^2 + x_2^2]^{1/2}}.
\]

Consider next the function

\[
F(x_1, x_2, x_3) = \frac{\partial E(x_1, x_2, x_3)}{\partial x_3} = -\frac{x_3}{4\pi r^3}.
\]

It is clear that for each \( x_3 \neq 0 \), \( F \) defines a regular distribution which depends parametrically on \( x_3 \). We want to calculate

\[
\lim_{x_3 \to 0^\pm} \langle F(x_1, x_2, x_3), \varphi(x_1, x_2) \rangle = \lim_{x_3 \to 0^\pm} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} -\frac{x_3}{4\pi r^3} \varphi(x_1, x_2) dx_1 \, dx_2.
\]

By introducing polar coordinates, we observe that

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{x_3}{2\pi r^3} dx_1 \, dx_2 = \int_{0}^{\infty} \frac{x_3 \rho \, d\rho}{(\rho^2 + x_3^2)^{3/2}} = 1, \quad x_3 > 0,
\]

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and that
\[ \lim_{x_3 \to 0 \pm} \int_{\rho_0}^{\infty} \frac{x_3 \rho \, d\rho}{(\rho^2 + x_3^2)^{3/2}} = 0, \text{ for any fixed } \rho_0 > 0. \]

It therefore follows from Exercise 5.4 (or 5.2) that \( x_3/2\pi r^3 \) is a \( \delta \) sequence and hence
\[ \lim_{x_3 \to 0^+} F(x_1, x_2, x_3) = -\frac{1}{2} \delta(x_1) \delta(x_2). \]

Since \( F \) is an odd function of \( x_3 \), we have
\[ \lim_{x_3 \to 0 \pm} F(x_1, x_2, x_3) = \mp \frac{1}{2} \delta(x_1) \delta(x_2). \quad (6.33) \]

Therefore, if \( \varphi(x_1, x_2) \) is any test function
\[ \lim_{x_3 \to 0 \pm} \int_{-\infty}^{\infty} -\frac{x_3}{4\pi r^3} \varphi(x_1, x_2) \, dx_1 \, dx_2 = \mp \frac{1}{2} \varphi(0, 0). \quad (6.34) \]

By a slight refinement of the arguments of Chapter 5, we can show that (6.34) holds for any bounded function \( \varphi \) which is continuous at \( x_1 = x_2 = 0 \). If we are interested in the behavior as \( x_3 \to 0 \) of tangential derivatives of \( E \) (that is, derivatives with respect to \( x_1 \) and \( x_2 \)), the situation is simpler. Take for instance \( \partial E / \partial x_1 \). Then, for \( x_3 \neq 0 \), we have, by (5.8),
\[ \left\langle \frac{\partial E}{\partial x_1}, \varphi \right\rangle = -\left\langle E, \frac{\partial \varphi}{\partial x_1} \right\rangle. \]

Now \( \partial \varphi / \partial x_1 \) is a test function in \( K_2 \), so that by (6.32),
\[ \lim_{x_3 \to 0 \pm} \left\langle \frac{\partial E}{\partial x_1}, \varphi \right\rangle = -\left\langle 1 \left/ 4\pi [x_1^2 + x_2^2]^{1/2} \right., \frac{\partial \varphi}{\partial x_1} \right\rangle. \]

In the same way,
\[ \lim_{x_3 \to 0 \pm} \left\langle \frac{\partial^{i+j} E}{\partial x_1^i \partial x_2^j}, \varphi \right\rangle = (-1)^{i+j} \left\langle 1 \left/ 4\pi [x_1^2 + x_2^2]^{1/2} \right., \frac{\partial^{i+j} \varphi}{\partial x_1^i \partial x_2^j} \right\rangle. \quad (6.35) \]

Thus every tangential derivative has well-defined distributional limits as \( x_3 \to 0 \pm \) and the limits from either side are equal.

We are now in a position to analyze the behavior of potentials of single and double layers spread on a flat surface \( \sigma \). We take \( \sigma \) to be part of the plane \( x_3 = 0 \) and assume that the origin \( x_1 = x_2 = x_3 = 0 \) is included in \( \sigma \). We first discuss the case of a simple layer of density \( a(x_1, x_2) \). The corresponding potential is given by
\[ u(x_1, x_2, x_3) = \int_{\sigma} a(\xi_1, \xi_2) \left\{ \frac{1}{4\pi[(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2 + x_3^2]^{1/2}} \right\} \, d\xi_1 \, d\xi_2. \]
For the purpose of investigating the behavior of \( u \) and its derivatives as \( x_3 \to 0 \) it is sufficient to study what happens in a neighborhood of the origin. The only drawback of such an analysis is that it is not valid on the edge \( C \) of \( \sigma \).

From (6.32) we find immediately (if \( a \) is continuous)

\[
\lim_{x_3 \to 0^\pm} u(0, 0, x_3) = \int_\sigma a(\xi_1, \xi_2) \frac{1}{4\pi[\xi_1^2 + \xi_2^2]^{1/2}} \, d\xi_1 \, d\xi_2, \tag{6.36}
\]

and, from (6.34),

\[
\lim_{x_3 \to 0^\pm} \frac{\partial u}{\partial x_3}(0, 0, x_3) = \lim_{x_3 \to 0^\pm} \int_\sigma a(\xi_1, \xi_2) \frac{-x_3}{4\pi[\xi_1^2 + \xi_2^2 + x_3^2]^{3/2}} \, d\xi_1 \, d\xi_2
\]

\[
= \mp \frac{1}{2} a(0, 0). \tag{6.37}
\]

Similarly, the limiting values of tangential derivatives are easily obtained using (6.35). Of course, one will then have to assume that appropriate derivatives of \( a(\xi_1, \xi_2) \) are continuous.

Next we consider the potential \( v \) of a double layer spread on the flat surface \( \sigma \). We have

\[
v(0, 0, x_3) = \int_\sigma b(\xi_1, \xi_2) \frac{\cos \theta}{4\pi r^2} \, d\xi_1 \, d\xi_2
\]

\[
= \int_\sigma b(\xi_1, \xi_2) \frac{x_3}{4\pi r^3} \, d\xi_1 \, d\xi_2,
\]

where \( r = [\xi_1^2 + \xi_2^2 + x_3^2]^{1/2} \). Using (6.34) and assuming that \( b \) is continuous, we find

\[
\lim_{x_3 \to 0^\pm} v(0, 0, x_3) = \pm \frac{1}{2} b(0, 0). \tag{6.38}
\]

Turning to \( \partial v / \partial x_3 \), we have

\[
\frac{\partial v}{\partial x_3} = \int_\sigma b(\xi_1, \xi_2) \left[ \frac{\xi_1^2 + \xi_2^2 - 2x_3^2}{4\pi r^5} \right] \, d\xi_1 \, d\xi_2. \tag{6.39}
\]

From (6.39) and the relations

\[
\frac{\partial^2}{\partial \xi_1^2} \left( \frac{1}{r} \right) = \frac{2\xi_1^2 - \xi_2^2 - x_3^2}{r^5}, \quad \frac{\partial^2}{\partial \xi_2^2} \left( \frac{1}{r} \right) = \frac{2\xi_2^2 - \xi_1^2 - x_3^2}{r^5},
\]

it follows that

\[
\frac{\partial v}{\partial x_3} = \int_\sigma b(\xi_1, \xi_2) \left( \frac{\partial^2}{\partial \xi_1^2} + \frac{\partial^2}{\partial \xi_2^2} \left( \frac{1}{4\pi r} \right) \right) \, d\xi_1 \, d\xi_2.
\]
Using Green’s theorem and assuming that $V^2 b$ is continuous, we find

$$\frac{\partial v}{\partial x_3} = \iiint_{\xi} \frac{1}{4\pi r} V^2 b \, d\xi_1 \, d\xi_2 + \int_C \left[ b \frac{\partial}{\partial p} \left( \frac{1}{4\pi r} \right) - \frac{\partial b}{\partial p} \frac{1}{4\pi r} \right] \, dC.$$ 

Here $C$ is the boundary of $\sigma$ and $p$ denotes the outward normal on $C$.

By (6.32) we can calculate the limiting value as $x_3 \to 0$ of the double integral. The integral over $C$ is continuous as $x_3 \to 0$, since the only singularity of the integrand is at $x_1 = x_2 = x_3 = 0$, which is not on the path of integration. Therefore,

$$\lim_{x_3 \to 0^\pm} \frac{\partial v}{\partial x_3} (0, 0, x_3) = \iiint_{\xi} \frac{1}{4\pi(\xi_1^2 + \xi_2^2)^{1/2}} (V^2 b) d\xi_1 \, d\xi_2$$

$$+ \int_C \left[ b \frac{\partial}{\partial p} \left( \frac{1}{4\pi(\xi_1^2 + \xi_2^2)^{1/2}} \right) - \frac{\partial b}{\partial p} \frac{1}{4\pi(\xi_1^2 + \xi_2^2)^{1/2}} \right] \, dC.$$ 

(6.40)

The right side of (6.40) may be considered to be the definition of the finite part of the divergent integral

$$\iiint_{\sigma} b(\xi_1, \xi_2) \frac{1}{4\pi(\xi_1^2 + \xi_2^2)^{3/2}} \, d\xi_1 \, d\xi_2,$$

which would result from (6.39) by setting $x_3 = 0$.

We are now ready to study layers spread on a smooth, curved surface $\sigma$. One side of this surface will be considered as the positive side and the other as the negative side. By agreement, when we speak of the normal to $\sigma$ we mean the normal to the positive side of the surface. If $\sigma$ is a closed surface without a boundary, as, for instance, the surface of a sphere, it is usual to think of the positive side as being the exterior side of $\sigma$; the normal to $\sigma$ then points outward from $\sigma$.

We shall investigate the behavior of potentials of layers in a neighborhood of a fixed point $s$ on $\sigma$. The normal at $s$ will be called $v$, whereas the normal at some arbitrary point $\xi$ on $\sigma$ will be denoted by $n$. Our procedure will start with dividing the surface $\sigma$ into two parts $\sigma_e$ and $\sigma - \sigma_e$, where $\sigma_e$ is the part of $\sigma$ within a sphere of radius $e$ with center at $s$ and $\sigma - \sigma_e$ is the remainder of $\sigma$.

Now consider any field quantity (that is, the potential or any of its derivatives) arising from the charges on $\sigma$. Such a field quantity is the sum of contributions from the charges on $\sigma - \sigma_e$ and on $\sigma_e$. As far as the charges on $\sigma - \sigma_e$ are concerned, $s$ is in a charge-free region and the contribution from $\sigma - \sigma_e$ is therefore continuous at $s$. Any discontinuity in a field quantity at $s$ must then be due to the contribution of the charges on $\sigma_e$. By choosing $e$ sufficiently small, we may regard $\sigma_e$ as a flat surface and we can then use

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previous results for plane surfaces to determine the behavior as the observation point \( x \) approaches \( s \).

For a simple layer we have, from (6.30),

\[
u(x) = \int_{\sigma - \sigma} a(\xi) \frac{1}{4\pi|x - \xi|} \, dS_\xi + \int_{\sigma} a(\xi) \frac{1}{4\pi|x - \xi|} \, dS_\xi.
\]

As \( x \to s \), the first integral is continuous, whereas, by (6.36), the second integral approaches:

\[
\int_{\sigma} a(\xi) \frac{1}{4\pi|s - \xi|} \, dS_\xi,
\]

which is a convergent integral over an infinitesimal region, so that

\[
\lim_{\varepsilon \to 0} \int_{\sigma} a(\xi) \frac{1}{4\pi|s - \xi|} \, dS_\xi = 0,
\]

\[
\lim_{x \to s} u(x) = \int_{\sigma} a(\xi) \frac{1}{4\pi|s - \xi|} \, dS_\xi. \tag{6.41}
\]

We note that the right side is just the value \( u(s) \) of the simple layer potential at a point on the surface [we have already established after (6.30) that the integral has meaning]. Thus the potential of a simple layer defined by (6.30) is continuous everywhere.

Let us now turn to the behavior of the normal derivative of the simple layer potential \( u \). With \( v \) denoting the normal to \( \sigma \) at the fixed point \( s \) on \( \sigma \), we wish to calculate

\[
\lim_{x \to s} \frac{\partial u}{\partial v}(x).
\]

If \( x \) is not on \( \sigma \), we can perform the differentiation under the integral sign in (6.30) to obtain

\[
\frac{\partial u}{\partial v}(x) = \int_{\sigma - \sigma} a(\xi) \frac{\cos(\xi - x, \nu)}{4\pi|\xi - x|^2} \, dS_\xi + \int_{\sigma} a(\xi) \frac{\cos(\xi - x, \nu)}{4\pi|\xi - x|^2} \, dS_\xi. \tag{6.42}
\]

As \( x \to s \), the first integral is continuous. The second integral is taken over a nearly flat surface, and to the order of approximation needed we may regard \( \sigma \) as lying in the tangent plane to \( \sigma \) at \( s \). We shall introduce the Cartesian coordinate system \((x_1, x_2, x_3)\) with origin at \( s \); here \( x_1 \), and \( x_2 \) are in the tangent plane and \( x_3 \) is in the normal direction (that is, \( x_3 \) is positive in the \( v \) direction and negative in the \( -v \) direction). If we let \( x \) lie on the normal to \( \sigma \) at \( s \), the second integral in (6.42) may be rewritten

\[
\int_{\sigma} a(\xi_1, \xi_2) \left[ -\frac{x_3}{4\pi(\xi_1^2 + \xi_2^2 + x_3^2)^{3/2}} \right] \, d\xi_1 \, d\xi_2.
\]
If we let \( x \rightarrow s \pm \), that is, \( x_3 \rightarrow 0 \pm \), we find from (6.37) that the integral approaches \( \mp \frac{1}{2} a(0, 0) \) or \( \mp \frac{1}{2} a(s) \). Thus we have, from (6.42),

\[
\lim_{x \rightarrow s \pm} \frac{\partial}{\partial v} u(x) = \mp \frac{1}{2} a(s) + \int_{\sigma} a(\xi) \frac{\cos(\xi - s, v)}{4\pi|s - \xi|^2} \, dS_\xi. \tag{6.43}
\]

We observe that (6.43) contains the well-known result of electrostatics that, at a charged surface, the normal component of the electric field jumps by an amount equal to the surface charge density. Of course, (6.43) also yields information on one-sided limits, so that the amount of effort we have expended is justified by the stronger conclusions reached.

By similar arguments one can show that any tangential derivative of \( u \) has equal limiting values as \( x \) approaches \( s \pm \). This common limiting value can be written as a surface integral involving the same tangential derivative of \( a(\xi_1, \xi_2) \).

We now turn to the potential of a double layer, which we write from (6.31) as

\[
v(x) = \int_{\sigma - e_\xi} b(\xi) \frac{\cos(x - \xi, n)}{4\pi|x - \xi|^2} \, dS_\xi + \int_{\sigma e} b(\xi) \frac{\cos(x - \xi, n)}{4\pi|x - \xi|^2} \, dS_\xi,
\]

where \( n \) is the normal to \( \sigma \) at \( \xi \). The first integral is continuous as \( x \rightarrow s \). The second integral is taken over a surface which is nearly flat. Thus in this integral we can identify the normal at \( \xi \) with the normal \( v \) at \( s \). Therefore,

\[
\int_{\sigma e} b(\xi) \frac{\cos(x - \xi, n)}{4\pi|x - \xi|^2} \, dS_\xi = \int_{\sigma e} b(\xi_1, \xi_2) \left[ \frac{x_3}{4\pi(\xi_1^2 + \xi_2^2 + x_3^2)^{3/2}} \right] d\xi_1 \, d\xi_2.
\]

As \( x_3 \rightarrow 0 \pm \), we find

\[
\lim_{x \rightarrow s \pm} v(x) = \pm \frac{1}{2} b(s) + \int_{\sigma} b(\xi) \frac{\cos(s - \xi, n)}{4\pi|s - \xi|^2} \, dS_\xi. \tag{6.44}
\]

Thus the potential of a double layer experiences a discontinuity upon crossing the layer, a fact which is again familiar from electrostatics.

For a given smooth surface \( \sigma \), \( \cos(s - \xi, n)/4\pi|s - \xi|^2 \) is a well-defined function of the two variables \( s \) and \( \xi \) which lie on \( \sigma \). Writing

\[
k(s, \xi) = \frac{\cos(s - \xi, n)}{4\pi|s - \xi|^2}, \tag{6.45}
\]

(6.44) becomes

\[
\lim_{x \rightarrow s \pm} v(x) = \pm \frac{1}{2} b(s) + \int_{\sigma} k(s, \xi) b(\xi) dS_\xi. \tag{6.46}
\]

We can also express the function \( \cos(\xi - s, v)/[4\pi|s - \xi|^2] \) appearing in (6.43) in terms of \( k \) defined by (6.45). In fact, a glance at the definitions shows that

\[
\frac{\cos(\xi - s, v)}{4\pi|s - \xi|^2} = k(\xi, s).
\]
Thus we can write (6.43) in a form similar to (6.46),

$$\lim_{x \to s^\pm} \frac{\partial}{\partial v} u(x) = \mp \frac{1}{2} a(s) + \int_{\sigma} k(\xi, s) a(\xi) dS_\xi. \quad (6.47)$$

It remains to discuss the behavior of $\partial v/\partial v$, the normal derivative of the double layer potential.

From the definition of $v$, we have, for $x$ on $\sigma$,

$$\frac{\partial v}{\partial x_3} = \frac{\partial v}{\partial x_3} = \int_{\sigma - \varepsilon} b(\xi) \frac{\partial}{\partial x_3} \left[ \frac{\cos \theta}{4\pi|x - \xi|^2} \right] dS + \int_{\sigma} b(\xi) \frac{\partial}{\partial x_3} \left[ \frac{\cos \theta}{4\pi|x - \xi|^2} \right] dS. \quad (6.48)$$

Again the first integral is continuous as $x \to s$; the second integral is taken over a flat surface, so that it can be written from (6.39) as

$$\int_{\sigma} b(\xi_1, \xi_2) \left[ \frac{\xi_1^2 + \xi_2^2 - 2x_3^2}{4\pi r^2} \right] d\xi_1 d\xi_2.$$

As $x_3 \to 0^\pm$, we find from (6.40) that this integral approaches

$$\int_{\sigma} \frac{1}{4\pi[\xi_1^2 + \xi_2^2]^{1/2}} \left( \nabla^2 b \right) d\xi_1 d\xi_2$$

$$+ \int_{C_{\varepsilon}} \left[ b \frac{\partial}{\partial \rho} \left( \frac{1}{4\pi[\xi_1^2 + \xi_2^2]^{1/2}} \right) \frac{\partial b}{\partial \rho} \frac{1}{4\pi[\xi_1^2 + \xi_2^2]^{1/2}} \right] dC.$$

If

$$\nabla^2 b = \frac{\partial^2 b}{\partial \xi_1^2} + \frac{\partial^2 b}{\partial \xi_2^2}$$

is continuous, the double integral is infinitesimal and, from (6.48),

$$\lim_{x \to s^\pm} \frac{\partial v}{\partial v} = \lim_{\varepsilon \to 0} \left( \int_{\sigma - \varepsilon} \frac{\partial}{\partial v} \left[ \frac{\cos \theta}{4\pi|x - \xi|^2} \right] dS \right.$$

$$+ \int_{C_{\varepsilon}} \left[ b \frac{\partial}{\partial \rho} \left( \frac{1}{4\pi|s - \xi|} \right) \frac{\partial b}{\partial \rho} \frac{1}{4\pi|s - \xi|} \right] dC \bigg). \quad (6.49)$$

In general, neither of the integrals on the right side has limiting values but the combination does. We shall see, in particular examples, that the expression on the right side of (6.49) can sometimes be simplified.

**EXERCISES**

6.18 In two-dimensional potential theory the fundamental solution for free space is

$$E(x \mid \xi) = \frac{1}{2\pi} \log \frac{1}{|x - \xi|}.$$
Let $C$ be a smooth curve. A simple layer of line density $a(x)$ on $C$ gives rise to the potential
\[ u(x) = \int_C a(\xi)E(x \mid \xi)\,dl_\xi, \]
where $dl_\xi$ is an element of length on $C$ at $\xi$. Show that $u(x)$ is well defined even when $x$ is a point $s$ on $C$.

A double layer of line density $b(x)$ on $C$ gives rise to the potential
\[ v(x) = \int_C b(\xi)\frac{\cos(x - \xi, n)}{2\pi|x - \xi|}\,dl_\xi, \]
where $n$ is the normal to $C$ at $\xi$. Show that $v(x)$ is well defined even when $x$ is a point $s$ on $C$. In fact, you should prove that the integrand does not even have a singularity at $x = \xi$.

6.19 Let
\[ E(x_1, x_2) = \frac{1}{2\pi} \log \frac{1}{[x_1^2 + x_2^2]^{1/2}}. \]
This is the fundamental solution for two-dimensional potential theory corresponding to a source at the origin.

Show that in the distributional sense,
\[ \lim_{x_1 \to 0 \pm} E(x_1, x_2) = E(x_1, 0), \]
\[ \lim_{x_1 \to 0 \pm} \frac{\partial E}{\partial x_2} = \mp \frac{1}{2} \delta(x_1). \]

6.20 Using the results of Exercises 6.18 and 6.19, show that the potentials of simple and double layers have the properties
\[ \lim_{x \to s \pm} u(x) = \int_C \frac{1}{2\pi} \log \frac{1}{|s - \xi|} a(\xi)\,dl_\xi, \quad (6.50) \]
\[ \lim_{x \to s \pm} \frac{\partial u}{\partial v} (x) = \mp \frac{1}{2} a(s) + \int_C k(\xi, s) a(\xi)\,dl_\xi, \quad (6.51) \]
\[ \lim_{x \to s \pm} v(x) = \pm \frac{1}{2} b(s) + \int_C k(s, \xi) b(\xi)\,dl_\xi. \quad (6.52) \]

Here $v$ and $n$ are the normals to $C$ at $s$ and $\xi$, respectively, and $k(s, \xi)$ is defined from
\[ k(s, \xi) = \frac{\cos(s - \xi, n)}{2\pi|s - \xi|}. \quad (6.53) \]
6.5 INTEGRAL EQUATIONS OF POTENTIAL THEORY

Interior Dirichlet Problem—Classical Formulation

Let $R$ be a bounded open region with a smooth boundary $\sigma$. Consider the Dirichlet problem for Laplace’s equation:

$$-\nabla^2 w = 0, \quad x \text{ in } R; \quad w = f, \quad x \text{ on } \sigma. \quad (6.54)$$

Here we shall assume that $f$ is a given continuous function of position on $\sigma$.

The classical approach (which has the advantage of providing an existence proof for the Dirichlet problem) attempts to find the solution of (6.54) in the form of the potential of a double layer spread on $\sigma$. In terms of the unknown density $b(\xi)$ of the double layer, the solution would then have the form

$$w(x) = \int_{\sigma} \frac{\cos (x - \xi, n)}{4\pi|x - \xi|^2} b(\xi) dS_\xi, \quad (6.55)$$

where $n$ is the outward normal to $\sigma$ at $\xi$.

It is clear that (6.55) is harmonic in $R$ and will be the solution of (6.54) if $b(\xi)$ is chosen so that

$$\lim_{x \to s^-} w(x) = f(s), \quad \text{for every } s \text{ on } \sigma. \quad (6.56)$$

Let us see whether $b(\xi)$ can be so chosen. According to (6.46), we have

$$\lim_{x \to s^-} w(x) = -\frac{1}{2} b(s) + \int_{\sigma} k(s, \xi) b(\xi) dS_\xi,$$

where

$$k(s, \xi) = \frac{\cos (s - \xi, n)}{4\pi|s - \xi|^2}$$

is in general a nonsymmetric kernel which generates a completely continuous integral operator. The proof of complete continuity is similar to the one used in Section 6.6. To satisfy (6.56), $b(\xi)$ must be a solution of the equation

$$f(s) = -\frac{1}{2} b(s) + \int_{\sigma} k(s, \xi) b(\xi) dS_\xi, \quad s \text{ on } \sigma, \quad (6.57)$$

which is a Fredholm equation of the second kind for $b(\xi)$. It can be shown that $\mu = \frac{1}{2}$ is not an eigenvalue of

$$\mu \varphi(s) = \int_{\sigma} k(s, \xi) \varphi(\xi) dS_\xi,$$

so that (6.57) must have one and only one solution. Thus (6.54) has one and only one solution of the form (6.55), with $b$ the unique solution of the
integral equation (6.57). By independent reasoning, we have previously shown that (6.54) has at most one solution; it therefore follows that (6.55) is the one and only solution of (6.54). In Exercises 6.21 and 6.22, (6.57) is actually solved in some particular cases and \( w \) is then calculated from (6.55).

The method just described provides an existence proof for the interior Dirichlet problem when \( \sigma \) is smooth and \( f \) continuous. Although these conditions can be somewhat relaxed, the method of integral equations fails entirely when \( \sigma \) is an arbitrary surface with no smoothness properties. In that case the existence of a solution of (6.54) must be established by more sophisticated methods, which we shall not explore here.

**Interior and Exterior Dirichlet Problems—Formulation as an Integral Equation of the First Kind**

A few preliminary remarks concerning the exterior Dirichlet problem are in order. Consider first the Dirichlet problem for the region exterior to the unit sphere in three dimensions. In addition to prescribing boundary values on the surface of the unit sphere, it will be necessary to impose some sort of boundary condition at infinity in order to get a unique solution. To illustrate the need for a boundary condition at infinity, we observe that the functions \( u_1(x) = 1 \) and \( u_2(x) = 1/r \) are both harmonic for \( r > 1 \) and assume the same value 1 on the surface of the unit sphere. In many physical applications the solution we are looking for should vanish at infinity and we will adopt this requirement in the general formulation of the exterior Dirichlet problem.

**Definition.** Let \( R_e \) be the exterior of a bounded region \( R_i \) in three dimensions, where the boundary of \( R_i \) is denoted by \( \sigma \). The *exterior Dirichlet problem* is the boundary value problem

\[
\nabla^2 u_e = 0, \quad x \text{ in } R_e; \quad u_e|_{\sigma} = f, \quad u_e|_{\infty} = 0. \tag{6.58}
\]

**Lemma 1.** The solution of (6.58) has the properties

\[
\left. u_e \right|_{\infty} = 0 \left( \frac{1}{r} \right), \quad \left. \frac{\partial u_e}{\partial r} \right|_{\infty} = 0 \left( \frac{1}{r^2} \right). \tag{6.59}
\]

**Proof.** According to Exercise 6.16, any harmonic function which vanishes at \( \infty \) can be written for sufficiently large \( r \) in the form

\[
\sum_{n=0}^{\infty} \sum_{m=-n}^{n} b_{mn} r^{-n-1} Y_n^m(\theta, \varphi),
\]

and the lemma then follows immediately. We note that the lemma holds for every choice of the origin of the coordinate system.

**Lemma 2.** Let \( E(x|\xi) = 1/4\pi|x - \xi| \) and let \( u_e \) satisfy (6.58). The point \( \xi \) is fixed and \( \sigma \), is the surface of a sphere of radius \( r \) with fixed arbitrary
\[
\lim_{r \to \infty} \int_{\partial \sigma} \left( E \frac{\partial u_e}{\partial r} - u_e \frac{\partial E}{\partial r} \right) dS_x = 0. \quad (6.60)
\]

**Proof.** We have
\[
E(x \mid \xi) = \frac{1}{4\pi(|\xi|^2 + r^2 - 2|\xi|r \cos \theta)^{1/2}} = \frac{1}{4\pi r} \left[ 1 + \frac{|\xi|^2}{r^2} - 2 \frac{|\xi|}{r} \cos \theta \right]^{1/2}
\]
where \( \theta \) is the angle between the vectors \( x \) and \( \xi \). Since \( \xi \) is fixed, it is clear from the binomial expansion that
\[
E(x \mid \xi) = 0 \left( \frac{1}{r} \right) \quad \text{as} \ r \to \infty, \quad \text{uniformly in} \ \theta.
\]
Similarly,
\[
\frac{\partial}{\partial r} E(x \mid \xi) = 0 \left( \frac{1}{r^2} \right) \quad \text{as} \ r \to \infty, \quad \text{uniformly in} \ \theta.
\]
Combining these estimates with (6.59), the integrand in (6.60) is \( O(1/r^3) \) uniformly in \( \theta \). Since the area of \( \sigma \), grows as \( r^2 \), it follows that (6.60) approaches 0 as \( r \to \infty \).

Simultaneously with (6.58) we shall consider the interior Dirichlet problem
\[
\nabla^2 u_i = 0, \quad x \in R_i; \quad u_i \mid_{\sigma} = f, \quad (6.61)
\]
where \( f \) is the same function that appears in (6.58).

The free-space fundamental solution \( E \) satisfies
\[
-\nabla^2 E = \delta(x - \xi), \quad \text{for all} \ x \ \text{and} \ \xi. \quad (6.62)
\]

Multiply (6.61) by \( E, (6.62) \) by \( u_i \), add, and integrate over \( R_i \) to obtain
\[
\int_{R_i} (E \nabla^2 u_i - u_i \nabla^2 E) dx = \begin{cases} u_i(\xi), & \xi \in R_i; \\ 0, & \xi \in R_e. \end{cases}
\]

By applying Green's theorem, this becomes
\[
\int_{\sigma} \left[ E(x \mid \xi) \frac{\partial u_i}{\partial n} - u_i \frac{\partial E(x \mid \xi)}{\partial n} \right] dS_x = \begin{cases} u_i(\xi), & \xi \in R_i; \\ 0, & \xi \in R_e, \end{cases} \quad (6.63)
\]
where \( n \) is the outward normal to \( R_i \) on \( \sigma \).

Similarly, we multiply (6.58) by \( E, (6.62) \) by \( u_e \), add, and integrate over the region bounded internally by \( \sigma \) and externally by a sphere \( \sigma_r \). After applying Green's theorem we note that the contribution from \( \sigma_r \) vanishes as \( r \to \infty \) by Lemma 2. Since the outward normal to \( R_e \) on \( \sigma \) is in the \(-n\) direction, we obtain
\[
\int_{\sigma} \left( -E \frac{\partial u_e}{\partial n} + u_e \frac{\partial E}{\partial n} \right) dS_x = \begin{cases} 0, & \xi \in R_i; \\ u_e(\xi), & \xi \in R_e. \end{cases} \quad (6.64)
\]

We now add (6.63) and (6.64). Since \( u_i \) and \( u_e \) are both equal to \( f \) on \( \sigma \), their
contributions cancel. Introducing the notation

\[ I(x) = \frac{\partial u_i}{\partial n} - \frac{\partial u_e}{\partial n}, \quad x \text{ on } \sigma, \]

we have

\[ \int_\sigma E(x, \xi)I(x)dS_x = \begin{cases} u_i(\xi), & \xi \text{ in } R_i; \\ u_e(\xi), & \xi \text{ in } R_e. \end{cases} \]

Using the symmetry of \( E \) and relabeling the variables, we obtain

\[ \int_\sigma E(x, \xi)I(\xi)dS_\xi = \begin{cases} u_i(\xi), & x \text{ in } R_i; \\ u_e(\xi), & x \text{ in } R_e. \end{cases} \tag{6.65} \]

Here of course \( I(\xi) \) is unknown, but it is remarkable that both \( u_i \) and \( u_e \) can be expressed as a simple layer potential. We have already seen in (6.55) that \( u_i \) can be expressed as a double layer potential. It remains to calculate \( I(\xi) \). The best we can do in general is to find an integral equation for \( I \) on \( \sigma \). If we let \( x \) approach a point \( s \) on the boundary in (6.65) and use the fact that potentials of simple layers are continuous, we find

\[ \int_\sigma \frac{1}{4\pi|s - \xi|} I(\xi)dS_\xi = f(s), \quad s \text{ on } \sigma. \tag{6.66} \]

Equation (6.66) is a Fredholm equation of the first kind over the surface \( \sigma \) for the unknown \( I(\xi) \). After solving for \( I \), we would substitute \( I \) in (6.65) to calculate \( u_i(x) \) and \( u_e(x) \).

From our discussion of integral equations of the first kind in Chapter 3, we know that (6.66) cannot have an \( L_2 \) solution for every \( f(s) \). The reason for the difficulty is that the quantities \( \partial u_i/\partial n \) and \( \partial u_e/\partial n \) do not necessarily exist in the ordinary sense for every Dirichlet problem. For instance, if \( f \) is a discontinuous function, we expect the gradient of \( u \) to have singularities on \( \sigma \) at the points where \( f \) is discontinuous. Remarkably enough our procedure works even in these cases if appropriate interpretations are made. In explicit cases it will turn out that the formal solution for \( I(\xi) \) is a series which diverges in the sense of ordinary convergence, but converges in the sense of distributions. When we then substitute back in (6.65) the integral will converge, even in the sense of ordinary convergence and provide the desired functions \( u_i \) and \( u_e \). Unfortunately, we will be unable to prove all these statements and the method is therefore only formal, but nevertheless, useful. In fact, when dealing with the Dirichlet problem for the exterior of a thin disk, a procedure similar to the one described here must be used.

**Example**

To illustrate the use of (6.66) and (6.65), consider the Dirichlet problem for the unit sphere. The integral equation (6.66) becomes

\[ f(\theta, \varphi) = \int_0^{2\pi} d\varphi' \int_0^\pi d\theta' \frac{\sin \theta'}{4\pi|s - \xi|} I(\theta', \varphi'), \tag{6.67} \]
where, in spherical coordinates,
\[ \xi = (1, \theta', \varphi'), \quad s = (1, \theta, \varphi). \]

Let \( \gamma \) be the angle between the vectors \( \xi \) and \( s \). According to Appendix A, equation (A.12), we can write formally
\[
\frac{1}{|s - \xi|} = \sum_{n=0}^{\infty} P_n(\cos \gamma) = \sum_{n=0}^{\infty} N_{0,n} \sum_{m=-n}^{n} \frac{Y_n^m(\theta, \varphi) \overline{Y}_n^m(\theta', \varphi')}{N_{m,n}}.
\]

If we then expand \( f \) and \( J \) in spherical harmonics, we can solve (6.67) and find
\[
I_{m,n} = \frac{4\pi f_{m,n}}{N_{0,n}}.
\]

To obtain \( u_i \) and \( u_e \), we substitute in (6.65) and make use of (A.8). This yields
\[
\begin{align*}
    u_i &= \sum_{n=0}^{\infty} \sum_{m=-n}^{n} r^2 f_{m,n} Y_n^m(\theta, \varphi) \\
    u_e &= \sum_{n=0}^{\infty} \sum_{m=-n}^{n} r^{-n-1} f_{m,n} \overline{Y}_n^m(\theta, \varphi).
\end{align*}
\]

These results agree with those obtained by separation of variables (Exercises 6.15 and 6.16). Note also that, although the series for \( u_i \) and \( u_e \) have meaning even when \( f \) is discontinuous, the series for \( I \) may diverge. This is the case, for instance, when
\[
f = \begin{cases} 
    1, & 0 < \theta < \pi/2; \\
    0, & \pi/2 < \theta < \pi.
\end{cases}
\]

### Neumann Problem

In the Neumann problem, the normal derivative is prescribed on \( \sigma \). Let \( R_t \) be a bounded open region with the smooth boundary \( \sigma \); the interior Neumann problem for Laplace's equation is
\[
-\nabla^2 u_i = 0, \quad x \text{ in } R_t; \quad \frac{\partial u_i}{\partial n} = f, \quad x \text{ on } \sigma. \tag{6.68}
\]

Here \( n \) is the outward normal to \( R \) and \( f \) is a given continuous function on \( \sigma \).

We first observe that (6.68) cannot have a solution for every \( f \)! This is clear from the physical interpretation of (6.68) as a steady-state heat-conduction problem. We have no sources in the region \( R_i \) and the heat flow is prescribed on \( \sigma \). These conditions are consistent with the steady state only if the total heat flow through \( \sigma \) vanishes. Thus we can expect that (6.68) will have a solution only if
\[
\int_{\sigma} f(\xi) dS_{\xi} = 0. \tag{6.69}
\]

That (6.69) is a necessary condition for a solution of (6.68) to exist follows.
directly from the fact that \( u_i \) is harmonic in \( R_i \). This implies that

\[
\int_{R_i} (\nabla^2 u_i) \, dx = 0.
\]

Since \( \nabla^2 = \text{div grad} \), we find, by using the divergence theorem, that

\[
\int_{\sigma} \frac{\partial u_i}{\partial n} \, dS = 0,
\]

which is just (6.69).

If condition (6.69) is satisfied, it can be shown that (6.68) has a solution, but that the solution is no longer unique. It is quite clear that to any particular solution of (6.68) we can add an arbitrary constant and the resulting function still satisfies (6.68). Just as for the Dirichlet problem, we can reduce (6.68) to the solution of an integral equation, but we shall not carry out the details.

For the exterior Neumann problem in three dimensions no such restriction as (6.69) is needed. The problem

\[
\nabla^2 u_e = 0, \quad x \text{ in } R_e; \quad \frac{\partial u_e}{\partial n} = f, \quad x \text{ on } \sigma; \quad u_e \big|_{\infty} = 0, \quad (6.70)
\]

has one and only one solution.

**Example 1**

The Neumann problem for the interior of the unit sphere in three dimensions. The boundary value problem is

\[
\nabla^2 u_i = 0, \quad r < 1; \quad \left( \frac{\partial u_i}{\partial r} \right)_{r=1} = f(\theta, \varphi).
\]

As we shall see below, the condition (6.69) will have to be imposed on \( f \). Any function harmonic in \( r < 1 \) can be represented in the form

\[
u_i(r, \theta, \varphi) = \sum_{n=0}^{\infty} \sum_{m=\pm n} a_{m,n} r^n Y_n^m(\theta, \varphi),\]

as was shown in Exercise 6.15. Setting

\[
f = \sum_{n=0}^{\infty} \sum_{m=\pm n} f_{m,n} Y_n^m(\theta, \varphi),
\]

we find from the boundary condition on \( u_i \) that

\[na_{m,n} = f_{m,n}, \quad n = 0, 1, 2, \ldots; \quad |m| \leq n.
\]

A solution for \( a_{0,0} \) is possible only if \( f_{0,0} = 0 \) [which is just condition (6.69)]. Then

\[a_{m,n} = \frac{f_{m,n}}{n}, \quad n \neq 0,
\]

and \( a_{0,0} \) is arbitrary.
The corresponding solution \( u_i \) is given by
\[
  u_i(r, \theta, \varphi) = A + \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \frac{1}{n} f_{m,n} r^n Y_n^m(\theta, \varphi),
\]
where \( A \) is an arbitrary constant.

**Example 2**

The Neumann problem for the exterior of the unit sphere in three dimensions. The boundary value problem is
\[
  \nabla^2 u_e = 0, \quad r > 1; \quad \left( \frac{\partial u_e}{\partial r} \right)_{r=1} = f(\theta, \varphi); \quad u_e \big|_{r=\infty} = 0. \tag{6.71}
\]
By Exercise 6.16,
\[
  u_e(r, \theta, \varphi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} b_{m,n} r^{-n-1} Y_n^m(\theta, \varphi).
\]
The boundary condition on \( u_e \) yields
\[
  (-n-1)b_{m,n} = f_{m,n}, \quad n = 0, 1, 2, \ldots; \quad |m| \leq n.
\]
We observe that we can solve for every \( b_{m,n} \). In fact,
\[
  b_{m,n} = -\frac{f_{m,n}}{n+1}.
\]
The corresponding solution of (6.71) is unique and no restriction on \( f \) is needed.
We have
\[
  u_e(r, \theta, \varphi) = -\sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{f_{m,n}}{n+1} r^{-n-1} Y_n^m(\theta, \varphi).
\]

**Dirichlet and Neumann Problems in Two Dimensions**

The interior Dirichlet and Neumann problems in two dimensions are amenable to the same analysis used in three dimensions (see, for instance Exercise 6.22). On the other hand, the formulation of the exterior problems requires some essential modifications.
Consider first the Dirichlet problem for the exterior of the unit circle. By separation of variables in polar coordinates, one finds easily that any function harmonic in \( r > 1 \) has the form
\[
  u(r, \varphi) = A + B \log r + \sum_{n=-\infty}^{\infty} (a_n r^{-|n|} + b_n r^{|n|}) e^{in\varphi}.
\]
If we require \( u \) to vanish at infinity, \( A, B, \) and all \( b_n \) must all vanish. This does not leave us with enough freedom to satisfy the boundary condition
\[ u(1, \varphi) = f(\varphi) \]  
On the other hand, if we retain the constant term \( A \) and relabel it \( a_0 \), we can choose \( a_n = f_n \) (where \( f_n \) is the \( n \)th Fourier coefficient of \( f \)) and the boundary condition at \( r = 1 \) is satisfied.

We are therefore led to the following formulation of the Dirichlet problem for the region \( R_e \) exterior to the closed curve \( C \):

\[
\nabla^2 u = 0, \quad x \text{ in } R_e; \quad u|_C = f, \quad u \text{ bounded at } \infty.
\]

The difference between two and three dimensions is that in the former we no longer require that \( u \) vanish at infinity but only that \( u \) be bounded at infinity. Although we shall not prove this, the problem has one and only one solution.

As we shall see in Section 6.8, not every exterior Dirichlet problem of physical origin falls in the category described above. For instance, the problem of a charged cylindrical conductor of cross section \( C \) leads to a different kind of Dirichlet problem.

Turning to the Neumann problem for the region \( R_e \) exterior to the closed curve \( C \), we look for a reasonable guide in formulating the boundary condition at infinity. The solution \( u \) is harmonic outside \( C \) and \( \partial u / \partial n = f \) on \( C \), where \( f \) is an arbitrary, prescribed function. Let \( C_r \) be any circle enclosing \( C \); the fact that \( u \) is harmonic in the annular region between \( C \) and \( C_r \) yields

\[
\int_{C_r} \frac{\partial u}{\partial r} r \, d\varphi = \int_C \frac{\partial u}{\partial n} \, dl = \int_C f \, dl.
\]

The last integral is prescribed and does not in general vanish. Thus

\[
\lim_{r \to \infty} \int_{C_r} \frac{\partial u}{\partial r} r \, d\varphi
\]

will not in general be 0.

Now any function harmonic for sufficiently large \( r \) and bounded at infinity has the representation

\[
v = \sum_{n = -\infty}^{\infty} a_n r^{-|n|} e^{in\varphi}.
\]

But it is clear that for such a function

\[
\lim_{r \to \infty} \int_{C_r} \frac{\partial v}{\partial r} r \, d\varphi = 0.
\]

Thus the solution of the exterior Neumann problem does not in general have the term which yields a nonzero flux at infinity. The function \( A \log r \) is harmonic for \( r \neq 0 \) and gives a nonzero flux. We are therefore led to the following formulation of the exterior Neumann problem:

\[
\nabla^2 u = 0, \quad x \text{ in } R_e; \quad \frac{\partial u}{\partial n} \bigg|_C = f, \quad \frac{u}{\log r} \text{ bounded at } \infty.
\]
It can then be shown that the exterior Neumann problem always has a solution, but that this solution is determined only to an additive constant.

**EXERCISES**

6.21 Consider the Dirichlet problem for the interior of the unit sphere in three dimensions. Show that \( \cos (s - \xi, n) = -|s - \xi|/2 \) and that (6.57) becomes

\[
f(s) = -\frac{1}{2} b(s) - \frac{1}{8\pi} \int_{\sigma} \frac{b(\xi)}{|s - \xi|} dS_{\xi}.
\]

Introduce spherical coordinates on \( \sigma \). Expand in spherical harmonics and use the formal relation

\[
\frac{1}{\sqrt{2 - 2 \cos \gamma}} = \sum_{n=0}^{\infty} P_n(\cos \gamma)
\]

to find \( b \). Substitute in (6.55) to obtain \( w \) and compare your result with (6.26) of Exercise 6.15.

6.22 Consider the inner Dirichlet problem in two dimensions. Using the equation (6.52) of Exercise 6.20, show that the solution can be written in the form

\[
w(x) = \int_{C} \frac{\cos (x - \xi, n)}{2\pi| x - \xi |} b(\xi) d\xi.
\]

For the case of the unit circle show that \( k(s, \xi) \) is a constant. Solve the integral equation by a Fourier series expansion. Find \( w \) and reduce it to (6.11).

6.23 By separation of variables, solve the interior and exterior Neumann problems for the unit circle. Show clearly that condition (6.69) is needed for the interior problem.

6.6 **GREEN'S FUNCTION FOR THE NEGATIVE LAPLACIAN**

**Definition and Properties**

Let \( R \) be an open, bounded region in \( n \)-dimensional space and let its boundary be denoted by \( \sigma \). The Green's function for the negative Laplacian in \( R \) is the solution \( g(x \mid \xi) \) of the boundary value problem

\[
-\nabla^2 g = \delta(x - \xi), \quad x \text{ and } \xi \text{ in } R; \quad g = 0, \quad x \text{ on } \sigma. \tag{6.74}
\]

Unless otherwise specified, all differentiations are with respect to the coordinates of \( x \).

The boundary value problem (6.74) in three dimensions has a simple interpretation in electrostatics or in steady heat conduction. We can view
\(g(x \mid \xi)\) as the temperature at any point \(x\) in \(R\) due to a unit source located at \(\xi\), when the boundary temperature is required to vanish. We may also think of \(g\) as the electrostatic potential due to a unit charge at \(\xi\) when the boundary potential vanishes (which is the case, for instance, if \(\sigma\) is a grounded metallic shell).

System (6.74) also occurs with two independent variables instead of three. Such a problem may be intrinsically two-dimensional as in the static deflection of a plane membrane subject to a transverse unit force; more often, it arises as a degenerate version of a three-dimensional problem when the applied source and the geometric configuration are independent of one of the Cartesian coordinates. An example of the latter is the potential of an infinite line source, of unit line density, parallel to a cylindrical conductor at zero potential.

Returning to the electrostatic interpretation of \(g\) in three dimensions, we see that \(g\) is the sum of the potential of the unit source at \(\xi\) in free space and of the potential due to the charge induced on \(\sigma\) (this charge is present on \(\sigma\) because of the requirement that the potential vanish there). In any number of dimensions we can write

\[ g(x \mid \xi) = E(x \mid \xi) + v(x, \xi), \]  

where \(E\) is the free-space fundamental solution satisfying

\[ -\nabla^2 E = \delta(x - \xi), \quad \text{for all } x \text{ and } \xi, \]  

\[ \nabla^2 v = 0, \quad x \text{ in } R; \quad v = -E, \quad x \text{ on } \sigma. \]  

Thus the problem of finding \(g\) is reduced to that of finding a harmonic function \(v\) in \(R\), assuming certain special boundary values on \(\sigma\). We remind the reader that \(E\) was calculated in Chapter 5, where we found

\[ E = \begin{cases} 
\frac{1}{2\pi} \log \frac{1}{|x - \xi|}, & \text{in two dimensions;} \\
\frac{1}{4\pi |x - \xi|}, & \text{in three dimensions;} \\
\frac{C_n}{|x - \xi|^{n-2}}, & \text{in } n \text{ dimensions, } n \geq 3.
\end{cases} \]

We now derive some important properties of \(g\).

**Theorem 1.** The Green’s function exists and is unique.

**Proof.** We need to show that \(v\) exists and is unique. But this follows from the existence and uniqueness of the Dirichlet problem for Laplace’s equation with continuous boundary values.

**Theorem 2.** The Green’s function is symmetric; that is,

\[ g(x \mid \xi) = g(\xi \mid x). \]
Proof. Let \( g(x \mid \xi) \) and \( g(x \mid \eta) \) be the Green's functions for the region \( R \) corresponding to sources located at \( \xi \) and \( \eta \), respectively. Thus

\[
-\nabla^2 g(x \mid \xi) = \delta(x - \xi), \quad x, \xi \text{ in } R; \quad g = 0, \quad x \text{ on } \sigma.
\]

\[
-\nabla^2 g(x \mid \eta) = \delta(x - \eta), \quad x, \eta \text{ in } R; \quad g = 0, \quad x \text{ on } \sigma.
\]

We multiply the first differential equation by \( g(x \mid \eta) \), the second by \( g(x \mid \xi) \), subtract, and integrate over \( R \) to obtain

\[
\int_R [g(x \mid \xi)\nabla^2 g(x \mid \eta) - g(x \mid \eta)\nabla^2 g(x \mid \xi)]dx
\]

\[
= \int_R [g(x \mid \eta)\delta(x - \xi) - g(x \mid \xi)\delta(x - \eta)]dx
\]

\[
= g(\xi \mid \eta) - g(\eta \mid \xi).
\]

Using Green's theorem the integral on the left side becomes

\[
\int_{\sigma} \left[ g(x \mid \xi) \frac{\partial}{\partial n} g(x \mid \eta) - g(x \mid \eta) \frac{\partial}{\partial n} g(x \mid \xi) \right] dS_x,
\]

where \( n \) is the outward normal to \( \sigma \) at \( x \) and \( dS_x \) is a surface element on \( \sigma \) at \( x \). Since \( g(x \mid \xi) \) and \( g(x \mid \eta) \) vanish when \( x \) is on \( \sigma \), we find

\[
g(\eta \mid \xi) = g(\xi \mid \eta), \quad \xi, \eta \text{ in } R. \quad (6.78)
\]

Thus the Green's function is a symmetric function of its arguments. Physicists refer to (6.78) as the reciprocity principle. Its significance is clear—the potential at \( \eta \) due to a unit source at \( \xi \) is equal to the potential at \( \xi \) due to a unit source at \( \eta \).

Theorem 3. The Green's function is positive in \( R \).

Proof. Draw a small sphere \( R_\varepsilon \) of radius \( \varepsilon \) with center at the source point \( \xi \) (see Figure 6.3). The part of \( R \) excluding \( R_\varepsilon \) will be denoted by \( R - R_\varepsilon \). In \( R - R_\varepsilon \), \( g(x \mid \xi) \) is harmonic in \( x \). The boundary of \( R - R_\varepsilon \) consists of \( \sigma \) and

\[FIGURE 6.3\]

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the spherical surface \(|x - \xi| = \varepsilon\). On \(\sigma, g\) vanishes; since \(g\) is positively infinite at \(x = \xi\), \(g\) will be positive on \(|x - \xi| = \varepsilon\) if \(\varepsilon\) is chosen sufficiently small. By the maximum principle for harmonic functions, \(g\) must be strictly positive in \(R - R_\varepsilon\). Since \(R_\varepsilon\) can be made as small as we please, \(g\) is strictly positive in \(R\).

**Theorem 4.** In three or more dimensions,

\[
0 < g(x \mid \xi) < E(x \mid \xi), \quad x, \xi \in R. \tag{6.79}
\]

**Proof.** For each fixed \(\xi\), the function \(v(x, \xi)\) of (6.75) is harmonic in \(R\). Moreover, on \(\sigma, v\) takes on the negative values

\[
-\frac{C_n}{|x - \xi|^{n-2}}, \quad n \geq 3.
\]

By the maximum principle, it follows that \(v\) is strictly negative in \(R\). Combining with Theorem 3, we obtain (6.79).

The appropriate modification for the two-dimensional case is the subject of Exercise 6.24.

**Theorem 5.** The integral operator generated by \(g\) is completely continuous.

**Proof.** It is clear from (6.75) that we can write

\[
g(x \mid \xi) = \frac{a(x, \xi)}{|x - \xi|^{n-2}}; \quad n \geq 3,
\]

where \(a\) is bounded for \(x, \xi \in R\), say \(|a(x, \xi)| < C\).

Let us consider the integral operator \(G\) generated by \(g\); that is, for each function \(\varphi(x)\) in \(L_2(R)\), we define

\[
u(x) = \int_R g(x \mid \xi) \varphi(\xi) d\xi = G\varphi; \quad x \in R.
\]

The function \(g(x \mid \xi)\) has a singularity at \(x = \xi\) so that it is not apparent that \(u(x)\) is even defined. Actually, we show below that \(u\) is in \(L_2(R)\) and that the operator \(G\) is completely continuous.

For \(\eta > 0\), we write

\[
g(x \mid \xi) = h_\eta(x \mid \xi) + l_\eta(x \mid \xi),
\]

where

\[
h_\eta(x \mid \xi) = \begin{cases} g(x \mid \xi), & |x - \xi| \geq \eta; \\ 0, & |x - \xi| < \eta; \end{cases}
\]

\[
l_\eta(x \mid \xi) = \begin{cases} 0, & |x - \xi| \geq \eta; \\ g(x \mid \xi), & |x - \xi| < \eta. \end{cases}
\]

The integral operators corresponding to these kernels satisfy the equation

\[
G = H_\eta + L_\eta.
\]
The operator $H_\eta$ is clearly completely continuous since it is generated by a bounded kernel over a finite region. We now show that to each $\epsilon > 0$, we can choose $\eta$ sufficiently small so that $\|L_\eta\| < \epsilon$. Indeed, we have

$$\|L_\eta\phi\|^2 = \left| \int_{|x - \xi| < \eta} \frac{a(x, \xi)}{|x - \xi|^{n-2}} \phi(\xi) \, d\xi \right|^2 \leq C \left\{ \int_{|x - \xi| < \eta} \frac{\phi(\xi)}{|x - \xi|^{n-2}} \, d\xi \right\}^2.$$

Furthermore, by the Schwarz inequality,

$$\left( \int_{|x - \xi| < \eta} \frac{\phi(\xi)}{|x - \xi|^{n-2}} \, d\xi \right)^2 \leq \left( \int_{|x - \xi| < \eta} \frac{1}{|x - \xi|^{(n-2)/2}} \, d\xi \right)^2 \int_{|x - \xi| < \eta} \frac{\phi(\xi)^2}{|x - \xi|^{(n-2)/2}} \, d\xi.$$

The first integral on the right side is easily evaluated by introducing spherical coordinates:

$$\int_{|x - \xi| < \eta} \frac{1}{|x - \xi|^{n-2}} \, d\xi = \frac{S_n \eta^2}{2},$$

where $S_n$ is the surface area of the $n$-dimensional unit sphere. The second integral is made larger by integrating over $R$ instead of $|x - \xi| < \eta$. Therefore,

$$\|L_\eta\phi\|^2 \leq C^2 S_n \eta^2 \left( \int_R \frac{\phi(\xi)^2}{|x - \xi|^{n-2}} \, d\xi \right).$$

Let $d$ be the diameter of $R$, that is,

$$d = \max_{x, \xi \in R} |x - \xi|;$$

then

$$\int_R \frac{1}{|x - \xi|^{n-2}} \, dx \leq \int_{R_d} \frac{1}{|x - \xi|^{n-2}} \, dx,$$

where $R_d$ is a sphere of radius $d$ with center at $\xi$. This last integral is explicitly given by

$$\frac{S_n d^2}{2},$$

so that,

$$\|L_\eta\phi\| \leq C S_n \frac{\eta d}{2} \|\phi\|.$$

By choosing $\eta$ small enough, we clearly have

$$\|L_\eta\| < \epsilon.$$
Thus for each $\varepsilon > 0$, we can write the integral operator $G$ as the sum of a completely continuous operator $H_\eta$ and an operator $L_\eta$ of norm smaller than $\varepsilon$. By Theorem 3, Section 2.10, it follows that $G$ is completely continuous. Our proof is valid for $n \geq 3$ but is easily modified for $n = 2$ (see Exercise 6.25).

**Theorem 6.** For the cases $n = 2$ and $n = 3$ we can say more: $G$ is then a Hilbert-Schmidt operator.

**Proof.** We give the proof in three dimensions. Then

\[
\int_R \int_R |g|^2 \, dx \, d\xi \leq C^2 \int_R \int_R \frac{1}{|x - \xi|^2} \, dx \, d\xi,
\]

\[
\int_R \frac{1}{|x - \xi|^2} \, dx \leq \int_{R^d} \frac{1}{|x - \xi|^2} \, dx = 4\pi d,
\]

so that

\[
\int_R \int_R |g|^2 \, dx \, d\xi \leq C^2 4\pi dV,
\]

where $V$ is the volume of $R$. The right side of the last inequality is finite and $g$ is therefore a Hilbert-Schmidt kernel and $G$ a Hilbert-Schmidt operator.

**Solution of the Dirichlet Problem**

The Green’s function plays its principal role in the solution of the Dirichlet problem for the Poisson equation

\[-\nabla^2 u = q(x), \quad \text{in } R; \quad u = f, \quad \text{on } \sigma. \quad (6.80)\]

Here $q$ is a given function defined in the region $R$ (the *forcing function* or *source function*) and $f$ is a given function on the boundary $\sigma$.

To solve the boundary value problem (6.80) we use the Green’s function defined in (6.74). Multiply the differential equation for $g$ by $u$, the one for $u$ by $g$, subtract, and integrate over $R$. After using Green’s theorem and the boundary conditions on $g$ and $u$, we find

\[u(\xi) = \int_R g(x | \xi)q(x)\, dx - \int_\sigma f(x) \frac{\partial g(x | \xi)}{\partial n_x} \, dS_x.\]

The notation $\partial/\partial n_x$ reminds us that the differentiation is with respect to the coordinates of $x$.

If we interchange the labels $x$ and $\xi$ and use the symmetry of $g$, we finally obtain

\[u(x) = \int_R g(x | \xi)q(\xi)\, d\xi - \int_\sigma \frac{\partial g(x | \xi)}{\partial n_\xi} f(\xi)\, dS_\xi. \quad (6.81)\]
The formula (6.81) expresses the solution of (6.80) in terms of the interior sources $g$ and the boundary data $f$. We may think of two influence functions at work: $g(x|\xi)$ for interior sources and

$$I(x|\xi) = -\frac{\partial g(x|\xi)}{\partial n_\xi}$$

(6.82)

for boundary sources. It is obvious that if $g$ is known, $I$ can easily be calculated from (6.82). Later we shall show that the converse is true; that is, a knowledge of $I$ permits us to find $g$.

Formula (6.81) can be used to establish the continuous dependence of $u$ on $q$ and $f$, and, with suitable assumptions on $q$, $f$, and $\sigma$, one can verify that (6.81) actually satisfies (6.80), but we shall not carry out the details. The verification is somewhat facilitated by observing that each integral on the right side of (6.81) satisfies its own boundary value problem. The first integral is the solution of

$$-\nabla^2 u = q, \quad x \text{ in } R; \quad u = 0, \quad x \text{ on } \sigma,$$

whereas the second integral satisfies

$$-\nabla^2 u = 0, \quad x \text{ in } R; \quad u = f, \quad x \text{ on } \sigma.$$

This latter problem, which is the Dirichlet problem for Laplace's equation, then has the solution

$$u(x) = -\int_\sigma f(\xi) \frac{\partial g(x|\xi)}{\partial n_\xi} \, dS_\xi = \int_\sigma I(x|\xi)f(\xi)dS_\xi,$$

(6.83)

which should be compared with (6.11).

We pause to derive a property of $I(x|\xi)$ which will be useful to us later on. Consider the Dirichlet problem for Laplace's equation with boundary data $f = 1$ on $\sigma$. The solution is obviously $u \equiv 1$ in $R$. Substituting in (6.83), we find

$$\int_\sigma I(x|\xi)dS_\xi = 1, \quad \text{for every } x \text{ in } R.$$

(6.84)

This result is physically evident if we recall the electrostatic interpretation of the problem. From (6.82) we see that $I(x|\xi)$ is the negative of the charge density induced on $\sigma$ by the presence of a unit source at the point $x$ when $\sigma$ is a grounded conductor. The total charge induced on $\sigma$ should be $-1$, to cancel the unit positive charge at $x$. Thus (6.84) follows.

**Eigenvalue Problem for the Negative Laplacian**

Let $R$ be a bounded region in $n$-dimensional space; consider the eigenvalue problem

$$-\nabla^2 u = \lambda u, \quad x \text{ in } R; \quad u = 0, \quad x \text{ on } \sigma.$$

(6.85)
System $(6.85)$ is intimately related to time-dependent equations such as the wave equation, but also has a connection with potential theory. All the important properties of $(6.85)$ can be derived by using the Green's function for the system $(6.74)$, which arises from a static rather than dynamic problem.

For most values of the complex parameter $\lambda$, the only solution of the system $(6.85)$ is the function $u \equiv 0$ in $R$. Any exceptional number $\lambda$, for which $(6.85)$ has a nontrivial solution $u$, is known as an eigenvalue; the corresponding nontrivial solution $u$ is an eigenfunction. If to some eigenvalue $\lambda$, there correspond two or more independent eigenfunctions, we say that $\lambda$ is a degenerate eigenvalue. If there is only one independent eigenfunction corresponding to $\lambda$—that is, if every eigenfunction corresponding to $\lambda$ can be written in the form $cu_1$, where $u_1$ is a fixed function and $c$ is an arbitrary constant—then we shall call $\lambda$ a simple eigenvalue.

To establish some properties of the eigenvalues and eigenfunctions, it will be convenient to introduce the following inner product and norm on $R$:

$$
\langle u, v \rangle = \int_R u \overline{v} \, dx
$$

$$
\|u\| = \langle u, u \rangle^{1/2} = \left[ \int_R |u|^2 \, dx \right]^{1/2}.
$$

All functions $u$ under consideration are supposed to belong to $L^2(R)$; that is, $\|u\| < \infty$. As usual we shall say that $u$ and $v$ are orthogonal if $\langle u, v \rangle = 0$.

If we multiply the differential equation in $(6.85)$ by $\overline{u}$ and integrate over $R$, we find

$$
\|u\|^2 = -\int_R \overline{u} \nabla^2 u \, dx.
$$

Using the first form of Green's theorem and appealing to the boundary condition on $u$, we have

$$
\lambda \|u\|^2 = \int_R (\nabla u \cdot \nabla \overline{u}) \, dx = \int_R |\nabla u|^2 \, dx.
$$

The right side is nonnegative and $\|u\| \neq 0$ if $u$ is an eigenfunction; therefore $\lambda$ is real, nonnegative. Moreover $\lambda = 0$ implies $\nabla u \equiv 0$ in $R$, that is, $u$ constant in $R$. Since $u$ must vanish on $\sigma$, the constant is 0, so that $u$ is not an eigenfunction. Thus we have shown that all eigenvalues of $(6.85)$ are real and positive.

Next let $u$ and $v$ be eigenfunctions corresponding, respectively, to the eigenvalues $\lambda$ and $\mu$, where $\lambda \neq \mu$. Since $\lambda$ and $\mu$ are real,

$$
-\nabla^2 u = \lambda u; \quad u = 0, \quad x \text{ on } \sigma;
$$

$$
-\nabla^2 \bar{v} = \mu \bar{v}; \quad \bar{v} = 0, \quad x \text{ on } \sigma.
$$
Multiply the first differential equation by \( \tilde{v} \), the second by \( u \), subtract, and integrate over \( R \), to obtain

\[
(\lambda - \mu) \langle u, v \rangle = \int_R (u \nabla^2 \tilde{v} - \tilde{v} \nabla^2 u) \, dx.
\]

From Green's theorem and the boundary conditions on \( u \) and \( \tilde{v} \), we find

\[
(\lambda - \mu) \langle u, v \rangle = 0,
\]

and, since \( \lambda \neq \mu \),

\[
\langle u, v \rangle = 0.
\]

We conclude that eigenfunctions corresponding to different eigenvalues are orthogonal.

We would like to prove next that the eigenfunctions of (6.85) form a complete set. The Green's function satisfying (6.74) plays an essential role in the proof. If we view the right side of the differential equation in (6.85) as an inhomogeneous term, we can use (6.81) to obtain the eigenvalue integral equation

\[
u(x) = \lambda \int_R g(x | \xi) u(\xi) \, d\xi.
\]

Since \( \lambda = 0 \) is not an eigenvalue of (6.85) we can set \( \mu = 1/\lambda \), so that

\[
\mu u(x) = \int_R g(x | \xi) u(\xi) \, d\xi = Gu,
\]

(6.86)

where \( G \) is the integral operator generated by the kernel \( g(x | \xi) \).

We observe that \( \mu = 0 \) is not an eigenvalue of (6.86). In fact, \( Gu = z \) implies \( -\nabla^2 z = u \). Thus if \( Gu = 0 \), we have \( u = 0 \). We therefore conclude that (6.85) and (6.86) have the same eigenfunctions and that their eigenvalues are reciprocals of each other. We may then investigate (6.86) instead of (6.85). For the purpose of calculating eigenfunctions, (6.86) is usually more difficult to handle than (6.85), but there are many properties of the eigenfunctions and eigenvalues which are more readily established from (6.86). We have shown earlier in this section that \( G \) is a symmetric, completely continuous integral operator. We recall that in Chapter 3, we have shown that such an operator has eigenfunctions which can be chosen to form a complete orthonormal set over \( R \).

Thus we have the important results: The eigenfunctions of (6.86) and hence of (6.85) form a complete orthonormal set over \( R \).

In the following examples we calculate the eigenfunctions of (6.85) in some simple cases by separation of variables.
EXAMPLES

Example 1. \( R \) is the rectangle \( 0 < x_1 < a, \ 0 < x_2 < b \). System (6.85) then becomes

\[
- \frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} = \lambda u, \quad 0 < x_1 < a, \ 0 < x_2 < b; \\
u(0, x_2) = u(a, x_2) = 0, \quad 0 < x_2 < b; \\
u(x_1, 0) = u(x_1, b) = 0, \quad 0 < x_1 < a.
\]

(6.87)

We attempt to find eigenfunctions by separation of variables. Later we shall prove that all eigenfunctions are obtained by this procedure. Substituting \( u = X_1(x_1)X_2(x_2) \) in the differential equation, we obtain

\[
\frac{X''_1}{X_1} = - \frac{X''_2}{X_2} - \lambda.
\]

One side is a function of \( x_1 \) only, the other of \( x_2 \) alone. Since both sides are equal for all values of the independent variables \( x_1 \) and \( x_2 \) in the ranges \( 0 < x_1 < a \) and \( 0 < x_2 < b \), it follows that they must be the same constant \( -\mu \). Therefore,

\[
X''_1 + \mu X_1 = 0, \quad 0 < x_1 < a; \quad X_1(0) = X_1(a) = 0; \\
X''_2 + (\lambda - \mu) X_2 = 0, \quad 0 < x_2 < b; \quad X_2(0) = X_2(b) = 0.
\]

The first system has eigenvalues

\[
\mu_m = \frac{m^2 \pi^2}{a^2}, \quad m = 1, 2, \ldots,
\]

with corresponding normalized eigenfunctions (over \( 0 < x_1 < a \))

\[
X_{1m}(x_1) = \left( \frac{2}{a} \right)^{1/2} \sin \frac{m \pi x_1}{a}.
\]

With \( \mu \) so determined, we turn to the equation for \( X_2 \). Nontrivial solutions are possible only if the parameter \( \lambda - \mu \) is \( n^2 \pi^2/b^2 \). The corresponding solution for \( X_2 \) is \( (2/b)^{1/2} \sin (n \pi x_2/b) \). Therefore, the eigenvalues of (6.87) are

\[
\lambda_{m,n} = \frac{m^2 \pi^2}{a^2} + \frac{n^2 \pi^2}{b^2}; \quad m = 1, 2, \ldots, \ n = 1, 2, \ldots
\]

and the eigenfunctions (orthonormal over the rectangle \( 0 < x_1 < a, \ 0 < x_2 < b \)) are

\[
u_{m,n} = \left( \frac{4}{ab} \right)^{1/2} \sin \frac{m \pi x_1}{a} \sin \frac{n \pi x_2}{b}.
\]

Although we could use a single index to classify the eigenvalues and eigenfunctions, it is much easier to use a double-index notation.
The set \( \{u_{m,n}\} \) is the set of all separable eigenfunctions of (6.87). At first it is not apparent that there might not exist nonseparable eigenfunctions of (6.87). We now dispose of this question. First, we observe that \( u_{m,n} \) is the product of two complete one-dimensional orthonormal sets. By Lemma 1, Section 3.1, it follows that \( u_{m,n}(x_1, x_2) \) is a complete orthonormal set over the rectangle \( 0 < x_1 < a, 0 < x_2 < b \). If there existed a nonseparable eigenfunction \( u \), it would have to be orthogonal to all \( u_{m,n} \). But this is impossible, since \( \{u_{m,n}\} \) is complete.

**Example 2.** Let \( R \) be the unit circle. Using polar coordinates, system (6.85) becomes

\[
- \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) - \frac{1}{r^2} \frac{\partial^2 u}{\partial \varphi^2} = \lambda u, \quad r < 1, \quad -\pi < \varphi < \pi;
\]

\[
u(1, \varphi) = 0, \quad -\pi < \varphi < \pi. \tag{6.88}
\]

We shall see below that additional boundary conditions will have to be imposed to ensure that \( u(r, \varphi) \) and grad \( u \) are continuous in the entire circle.

We look for solutions of (6.88) that are separable in polar coordinates. Substituting \( u = R(r)\Phi(\varphi) \) in the differential equation, we find

\[
\frac{\Phi''}{\Phi} = -\frac{r(rR')' - \lambda r^2 R}{R}.
\]

By the same argument used in Example 1, we must have

\[
\Phi'' + \mu \Phi = 0, \quad -\pi < \varphi < \pi;
\]

\[
(rR')' + \left( \lambda r - \frac{\mu}{r} \right) R = 0, \quad 0 < r < 1.
\]

The set of points \( \varphi = \pi \) and \( \varphi = -\pi \) represent the same radial line in the unit circle. For \( R\Phi \) and grad \( R\Phi \) to be continuous in the circle, we need to impose the boundary conditions

\[
\Phi(-\pi) = \Phi(\pi), \quad \Phi'(-\pi) = \Phi'(\pi).
\]

It therefore follows that nontrivial solutions of the \( \varphi \) equation can be obtained only for

\[
\mu = n^2, \quad n = 0, 1, 2, \ldots
\]

The eigenvalue \( n = 0 \) is simple and has the eigenfunction \( \Phi_0 = \text{constant} \). The eigenvalue \( n^2, n \neq 0 \), is degenerate and has the independent eigenfunctions \( e^{in\varphi} \) and \( e^{-in\varphi} \). As a matter of convenience we shall think of \( n \) as taking all integral values (positive, negative, and zero); then

\[
\mu = n^2, \quad n = \ldots, -2, -1, 0, 1, 2, \ldots
\]
and the eigenfunctions are
\[ \Phi_n(\varphi) = e^{in\varphi}, \quad n = \ldots, -2, -1, 0, 1, 2, \ldots. \]
We observe that the set \( \Phi_n(\varphi) \) is an orthogonal set over \( -\pi < \varphi < \pi \); that is,
\[ \int_{-\pi}^{\pi} \Phi_n(\varphi)\overline{\Phi_m(\varphi)}d\varphi = 0, \quad m \neq n. \]
The normalized eigenfunctions are
\[ \left( \frac{1}{2\pi} \right)^{1/2} e^{in\varphi}. \]

Turning to the radial equation with \( \mu = n^2 \), we make the substitution \( z = \sqrt{\lambda} r \), which is permissible, since \( \lambda \) is positive. We then obtain the Bessel equation of order \( n \):
\[ \frac{d}{dz} \left( z \frac{dR}{dz} \right) - \frac{n^2}{z} R = 0, \quad n = \ldots, -2, -1, 0, 1, 2, \ldots. \]
This differential equation depends on the nonnegative parameter \( n^2 \) but not on \( n \) itself. The general solution bounded at the origin is \( CJ_{|n|}(\sqrt{\lambda} r) \). We have shown in Appendix B, Volume I, that \( J_{-n} = (-1)^n J_n \), so that we can as well write
\[ R(r) = \text{constant} \ J_n(\sqrt{\lambda} r), \quad n = \ldots, -2, -1, 0, 1, 2, \ldots \]
We must still impose the boundary condition at \( r = 1 \). This leads to the equation
\[ J_n(\sqrt{\lambda}) = 0 \]
to determine \( \lambda \). With \( n \) fixed, the equation has an infinite number of positive roots which we label
\[ \beta_1^{(n)}, \beta_2^{(n)}, \ldots. \]
Thus the eigenvalues of (6.88) are
\[ \lambda_{n,k} = [\beta_k^{(n)}]^2; \quad n = \ldots, -2, -1, 0, 1, 2, \ldots, \quad k = 1, 2, \ldots. \]
One should observe that, since \( J_{-n} \) is proportional to \( J_n \), \( \lambda_{n,k} = \lambda_{-n,k} \).
The corresponding eigenfunctions of (6.88) are
\[ u_{n,k} = e^{in\varphi} J_n(\beta_k^{(n)} r). \]
As was seen in equation (4.102), the functions \( J_n(\beta_k^{(n)} r) \) with \( n \) fixed, form an orthogonal set with weight \( r \) over \( 0 < r < 1 \). We also derived the normalization integral
\[ \int_0^1 r J_n^2(\beta_k^{(n)} r)dr = \frac{1}{2}[J'_n(\beta_k^{(n)})]^2. \]
Therefore, the set of functions

\[ u_{n,k} = e^{i\nu} \frac{e^{in\phi}}{(\pi)^{1/2} J_n'(\beta_k^n)} J_n(\beta_k^n r) \]

is orthonormal over the unit circle, that is,

\[ \int u_{n,k} \bar{u}_{m,j} dx = \int_0^\pi d\phi \int_0^1 r dr \ u_{n,k}(r, \phi) \bar{u}_{m,j}(r, \phi) \]

\[ = \begin{cases} 
0, & m \neq n \text{ or } j \neq k; \\
1, & m = n \text{ and } j = k.
\end{cases} \]

By arguments similar to those used in Example 1, we can show that the set \( u_{n,k} \) is complete and that there are no nonseparable eigenfunctions.

**Green's Function for Unbounded Regions**

We deal first with three-dimensional problems. The simplest kind of unbounded region is the exterior \( R_e \) of a bounded region \( R_i \) with boundary \( \sigma \). Suppose we think of \( \sigma \) as a grounded metallic shell (that is, at zero potential) and place a unit charge at the point \( \xi \) outside \( \sigma \). The potential \( g(x \mid \xi) \) will then satisfy the system

\[ -\nabla^2 g = \delta(x - \xi), \quad x, \xi \text{ in } R_e; \quad g = 0, \quad x \text{ on } \sigma. \]

In addition, a boundary condition must be imposed at infinity. To see what this boundary condition should be, we write

\[ g(x \mid \xi) = E(x \mid \xi) + v(x, \xi), \]

where \( E = 1/4\pi|x - \xi| \) is the free-space potential of the unit source (as if no metallic shell were present) and \( v \) is the potential due to the charges induced on \( \sigma \). This latter potential clearly vanishes at infinity since it is due to charges distributed in a finite portion of space. The function \( v \) satisfies the exterior Dirichlet problem

\[ \nabla^2 v = 0, \quad x \text{ in } R_e; \quad v = -E, \quad x \text{ on } \sigma; \quad v = 0, \quad \text{for } |x| = \infty. \tag{6.89} \]

We have seen in Section 6.5 that this problem has one and only one solution. Since \( E \) vanishes at \( |x| = \infty \), we can now formulate the problem for the exterior Green's function in its complete form as

\[ -\nabla^2 g = \delta(x - \xi), \quad x, \xi \text{ in } R_e; \quad g = 0, \quad x \text{ on } \sigma; \quad g = 0, \quad |x| = \infty. \tag{6.90} \]

The solution of the exterior Dirichlet problem for Poisson's equation can now be found in terms of \( g \). Let \( q(x) \) be defined in \( R_e \) and let \( q \) vanish outside some finite sphere. Consider the problem

\[ -\nabla^2 u = q, \quad x \text{ in } R_e; \quad u = f, \quad x \text{ on } \sigma; \quad u = 0, \quad |x| = \infty. \tag{6.91} \]

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To solve (6.91), multiply (6.90) by $u$, (6.91) by $g$, subtract, and integrate over a region bounded internally by $\sigma$ and externally by the surface $\sigma_r$ of a large sphere. When we apply Green's theorem contributions from $\sigma_r$ will vanish as $r \to \infty$ because of the behavior of $g$ and $u$ at infinity. We therefore find

$$u(x) = \int_{R_e} g(x \mid \xi)q(\xi)d\xi + \int_{\sigma} \frac{\partial g(x \mid \xi)}{\partial n_\xi} f(\xi)dS_\xi,$$

(6.92) which should be compared with (6.81). The reason for the change in sign in the last term is due to the fact that the normal to $R_e$ on $\sigma$ is in the $-n$ direction (the $n$ direction is outward from $R_d$).

The eigenvalue problem (6.85) changes character entirely when $R$ is an exterior region. It is no longer true that the integral operator $G$ generated by $g(x \mid \xi)$ is completely continuous. Theorem 5 is based on the fact that the region is bounded. In the language of Chapter 2, the spectrum of $G$ is no longer discrete but is now continuous. Related questions will be taken up when we deal with the wave equation in exterior domains.

So far we have restricted ourselves to a region exterior to a simple bounded region. If $R_e$ is exterior to a finite number of bounded regions (say the exterior of two nonoverlapping spheres) the analysis proceeds in the same way and no major changes are needed, on the other hand, if $R_e$ is a wedge-shaped region (neither a bounded region nor the exterior of one), we must make certain modifications. Such problems will be treated on an ad hoc basis.

**Exercises**

6.24 Consider a bounded region $R$ in two dimensions. Show that the appropriate form of (6.79) is

$$0 < g(x \mid \xi) < \frac{1}{2\pi} \log \frac{h}{|x - \xi|},$$

where $h$ is the diameter of $R$; that is,

$$h = \max_{x, \xi \text{ on } \sigma} |x - \xi|.$$

[Hint: Apply the maximum principle for harmonic functions to

$$v(x, \xi) - \frac{1}{2\pi} \log h.]$$

6.25 Show that, for a bounded region in two dimensions, $G$ is a Hilbert-Schmidt operator and hence completely continuous.

6.26 In $n$-dimensional space show that the kernel

$$\frac{a(x, \xi)}{|x - \xi|^m}, \quad m < n, \quad a \text{ continuous},$$

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generates a completely continuous integral operator on a bounded region $R$.

6.27 Prove that $g(x | \xi)$ in (6.90) is symmetric.

6.28 Find all eigenvalues and eigenfunctions of the system

$$\nabla^2 u = \lambda u, \quad x \text{ in } R; \quad u = 0, \quad x \text{ on } \sigma,$$

when $R$ is the sector

$$r < 1, \quad 0 < \varphi < \alpha.$$

Normalize the eigenfunctions.

6.29 Same as Exercise 6.28 for the region $R$ between two concentric circles.

6.30 Consider the same problem as Exercise 6.28 but now for the interior of the unit sphere in three dimensions. The boundary value problem in spherical coordinates becomes

$$-\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} Su = \lambda u, \quad u(1, \theta, \varphi) = 0, \quad (6.93)$$

where $S$ is the operator defined in (A.1).

If we write

$$u = R(r)Y(\theta, \varphi),$$

and label the separation constant $\mu$, we find

$$SY = \mu Y,$$

$$\frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) - \mu R + \lambda r^2 R = 0.$$

Since $Y$ and grad $Y$ must be continuous on an entire spherical surface, we find, as in Appendix A, that

$$\mu = n(n + 1), \quad n = 0, 1, 2, \ldots$$

$$Y = Y_n^m(\theta, \varphi) = P_n^m(\cos \theta)e^{im\varphi}, \quad |m| \leq n.$$

In the radial equation [with $\mu = n(n + 1)$] we first change the independent variable to $z = \sqrt{\lambda} r$ and then the dependent variable to $Z = \sqrt{\lambda} R$. This leads to

$$\frac{d}{dz} \left( z \frac{dZ}{dz} \right) + zZ - \frac{(n + 1/2)^2}{z} Z = 0,$$

which is Bessel's equation of order $n + 1/2$. Imposing the boundary condition $R(1) = 0$ [that is, $Z(\sqrt{\lambda}) = 0$] and requiring that the solution be finite at the origin, we find that the eigenvalues $\lambda$ are determined from

$$J_{n+(1/2)}(\sqrt{\lambda}) = 0, \quad n = 0, 1, 2, \ldots$$
With \( n \) fixed \( J_{n+(1/2)}(x) \) has an infinite number of positive roots labeled
\[ \beta_k^{(n+1/2)}, \quad k = 1, 2, \ldots. \]
Thus the eigenvalues of (6.93) are
\[ \lambda_{n,k} = [\beta_k^{(n+1/2)}]^2, \quad n = 0, 1, 2, \ldots, \quad k = 1, 2, \ldots \]
and the eigenfunctions are
\[
u_{m,n,k} = \frac{1}{r} J_{n+(1/2)}(\beta_k^{(n+1/2)}r) Y_n^m(\theta, \varphi).
\]
These functions are orthogonal over the unit sphere; that is,
\[
\int_0^1 r^2 dr \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\varphi \ u_{m,n,k} \bar{u}_{p,q,j} = \begin{cases} 0, & m \neq p \text{ or } n \neq q \text{ or } k \neq j; \\ N_{m,n,k}, & m = p, n = q, k = j. \end{cases}
\]
The calculation of \( N_{m,n,k} \) is left to the reader [see equation (A.2)].

6.31 Same as Exercise 6.28 for the finite region \( R \) bounded by the unit sphere and the cone \( \theta = \alpha \), where \( 0 < \alpha < \pi \).

6.32 Same as Exercise 6.28 for the finite region \( R \) bounded by the unit sphere and the planes \( \varphi = 0 \) and \( \varphi = \alpha \), where \( 0 < \alpha < 2\pi \).

6.33 For the bounded region \( R \), consider the eigenvalue problem
\[-\nabla^2 u = \lambda u, \quad x \in R; \quad \frac{\partial u}{\partial n} = 0, \quad x \text{ on } \sigma.
\]
(a) Show that the eigenvalues are real, nonnegative.
(b) Show that \( \lambda = 0 \) is an eigenvalue to which corresponds the constant eigenfunction.
(c) Find all eigenvalues and eigenfunctions when \( R \) is the interior of a circle of radius \( a \).

6.34 For the bounded region \( R \), consider the eigenvalue problem
\[-\nabla^2 u = \lambda u, \quad x \in R; \quad \frac{\partial u}{\partial n} + hu = 0, \quad x \text{ on } \sigma,
\]
where \( h \) is a positive given function on \( \sigma \).
(a) Show that the eigenvalues are real and positive, and that eigenfunctions corresponding to different eigenvalues are orthogonal.
(b) Solve the eigenvalue problem when \( h \) is a constant and \( R \) is the interior of the unit circle.
6.7 METHODS FOR DETERMINING THE GREEN'S FUNCTION

Let $R$ be a bounded domain; we want to find the Green's function $g(x \mid \xi)$ for the negative Laplacian in $R$. By definition $g(x \mid \xi)$ satisfies (6.74), repeated below for convenience

$$-\nabla^2 g(x \mid \xi) = \delta(x - \xi), \quad x, \xi \text{ in } R; \quad g = 0, \quad x \text{ on } \sigma. \quad (6.94)$$

Once we have found $g$ we can solve a variety of boundary value problems for the Laplacian operator in $R$ [see (6.81)]. It will turn out that $g$ can easily be calculated if we can find the values of the normal derivative of $g$ on the boundary. Anticipating this, we use the notation introduced in (6.82):

$$I(x \mid \xi) = -\frac{\partial g}{\partial n}(x \mid \xi),$$

where $x$ is an arbitrary point in $R$, $\xi$ is an arbitrary point on $\sigma$, and $n$ is the outward normal at $\xi$.

**Integral Equation Method**

Let $E(x \mid t)$ be the fundamental solution in free space. Then $E(x \mid t)$ satisfies

$$-\nabla^2 E = \delta(x - t), \quad (6.95)$$

and $E$ has been explicitly calculated [(5.104)]. To find $g$ we multiply the equation (6.94) by $E$, the equation (6.95) by $g$, subtract, and integrate over $R$ to obtain

$$E(\xi \mid t) - g(t \mid \xi) = \int_R [g(x \mid \xi)\nabla^2 E(x \mid t) - E(x \mid t)\nabla^2 g(x \mid \xi)]dx.$$

We now use Green's theorem, the boundary condition on $g$, and the symmetry of $E$ and $g$ to find, after a relabeling of the variables,

$$g(x \mid \xi) = E(x \mid \xi) - \int_\sigma E(x \mid t)I(\xi \mid t)dS; \quad x, \xi \text{ in } R. \quad (6.96)$$

If we compare this expression with (6.75), we see that the integral term is just the function previously called $-\nu(x, \xi)$. It is also clear from (6.96) that $g$ will be known once we have calculated $I(\xi \mid t)$. To find $I(\xi \mid t)$, we let $x$ approach the boundary from the interior. Since the integral in (6.96) is the potential of a simple layer on $\sigma$, we can use the fact that such layers give rise to a continuous potential to obtain

$$\lim_{x \to s} g(x \mid \xi) = E(s \mid \xi) - \int_\sigma E(s \mid t)I(\xi \mid t)dS; \quad \xi \text{ in } R, \quad s \text{ on } \sigma.$$
Now \( g(s \mid \xi) = 0 \), so that \( I(\xi \mid t) \) considered as a function of position on \( \sigma \) satisfies the integral equation
\[
E(s \mid \xi) = \int_{\sigma} E(s \mid t) I(\xi \mid t) dS_t, \quad s \text{ on } \sigma,
\]
or, after relabeling variables,
\[
E(s \mid x) = \int_{\sigma} E(s \mid \xi) I(x \mid \xi) dS_{\xi}, \quad s \text{ on } \sigma. \tag{6.97}
\]

Here the functions \( E(s \mid x) \) and \( E(s \mid \xi) \) are known explicitly. For each fixed \( x \) in \( R \), \( I(x \mid \xi) \) is a function of position on \( \sigma \) and satisfies the integral equation (6.97) on \( \sigma \). This is a Fredholm equation of the first kind; as we have seen in Chapter 3, the question of existence of a solution of such an equation is more difficult than for an equation of the second kind. We are not concerned here with questions of existence which can be answered satisfactorily by other methods; instead we want procedures for calculating \( g \) (or \( I \)) and for this purpose (6.97) is useful.

**Example**

Consider the Green's function for the unit circle. Using polar coordinates we take \( x = (r, 0) \), \( s = (1, \theta) \), \( \xi = (1, \psi) \) (see Figure 6.4). Then
\[
E(s \mid x) = -\frac{1}{2\pi} \log \left[ 1 + r^2 - 2r \cos \theta \right]^{1/2} = -\frac{1}{4\pi} \log(1 + r^2 - 2r \cos \theta),
\]
\[
E(s \mid \xi) = -\frac{1}{4\pi} \log \left[ 2 - 2 \cos (\psi - \theta) \right].
\]

We further observe that \( I(x \mid \xi) \) depends only on \( r \) and \( \psi \) and we therefore introduce the more convenient notation \( I(r, \psi) \). The integral equation (6.97) becomes
\[
\log \left( 1 + r^2 - 2r \cos \theta \right) = \int_{-\pi}^{\pi} \log \left[ 2 - 2 \cos (\psi - \theta) \right] I(r, \psi) d\psi, \quad -\pi < \theta < \pi.
\]
Now $I(r, \psi)$ is clearly an even function of $\psi$, so that we can expand $I$ in a cosine series

$$I(r, \psi) = \sum_{n=0}^{\infty} I_n(r) \cos n\psi.$$  

From (6.84) it follows that $I_0(r) = 1/2\pi$, and from (6.22) we have

$$\log (1 + r^2 - 2r \cos \theta) = -2 \sum_{n=1}^{\infty} \frac{r^n \cos n\theta}{n},$$

$$\log [2 - 2 \cos (\psi - \theta)] = -2 \sum_{n=1}^{\infty} \frac{\cos n(\psi - \theta)}{n}$$

$$= -2 \sum_{n=1}^{\infty} \frac{\cos n\psi \cos n\theta + \sin n\psi \sin n\theta}{n}.$$  

Substituting in the integral equation and using the orthogonality properties of trigonometric functions, we find

$$\sum_{n=1}^{\infty} \frac{r^n \cos n\theta}{n} = \pi \sum_{n=1}^{\infty} \frac{\cos n\theta}{n} I_n(r), \quad -\pi < \theta < \pi,$$

which implies

$$I_n(r) = \frac{1}{\pi} r^n, \quad n = 1, 2, \ldots.$$  

Combining this with the result for $I_0$, we have, by (6.21),

$$I(r, \psi) = \frac{1}{2\pi} + \frac{1}{\pi} \frac{r \cos \psi - r^2}{1 + r^2 - 2r \cos \psi}.$$  

Now if the point $x$ is taken as $(r, \varphi)$ instead of $(r, 0)$, we have

$$I = \frac{1}{2\pi} \frac{1 - r^2}{1 + r^2 - 2r \cos (\psi - \varphi)}.$$  

If we are interested only in solving the Dirichlet problem for Laplace's equation in the unit circle we do not need to calculate $g$. Indeed (6.83) yields

$$u(r, \varphi) = \int_{-\pi}^{\pi} f(\psi) \frac{1}{2\pi} \frac{1 - r^2}{1 + r^2 - 2r \cos (\psi - \varphi)} d\psi,$$

which is just the result obtained in (6.11).

For other purposes one might wish an explicit expression for $g(x | \xi)$. By (6.96),

$$g(r, \varphi | r_0, \varphi_0) = E(r, \varphi | r_0, \varphi_0) - \int_{-\pi}^{\pi} E(r, \varphi | 1, \psi)I(r_0, \varphi_0 | 1, \psi) d\psi. \quad (6.98)$$

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Now
\[ E(r, \varphi | 1, \psi) = -\frac{1}{4\pi} \log \left[ 1 + r^2 - 2r \cos (\varphi - \psi) \right] = \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{r^n \cos n(\varphi - \psi)}{n}, \]
\[ I(r_0, \varphi_0 | 1, \psi) = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{r_0^n \cos n(\varphi_0 - \psi)}{n}. \]
Substituting in (6.98), and availing ourselves of the orthogonality of trigonometric functions, we find
\[ g(r, \varphi | r_0, \varphi_0) = E(r, \varphi | r_0, \varphi_0) - \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{(rr_0)^n}{n} \cos n(\varphi - \varphi_0). \]
The sum can be evaluated by (6.22) to give
\[ g(r, \varphi | r_0, \varphi_0) = E(r, \varphi | r_0, \varphi_0) + \frac{1}{4\pi} \log \left[ 1 + (rr_0)^2 - 2rr_0 \cos (\varphi - \varphi_0) \right]. \]
(6.99)

Thus \( g \) is the sum of the free space Green's function and a function \( v(r, \varphi; r_0, \varphi_0) \) as in (6.75). The function \( v \) can be given a simple physical interpretation. Let \( x^*_0 \) be the inverse point of \( x_0 = (r_0, \varphi_0) \) with respect to the unit circle; that is, \( x^*_0 = (1/r_0, \varphi_0) \). The point \( x^*_0 \) is outside the circle and
\[ |x - x^*_0|^2 = r^2 + \frac{1}{r_0^2} - 2 \frac{r}{r_0} \cos (\varphi - \varphi_0) \]
\[ = \frac{1}{r_0^2} \left[ 1 + r^2r_0^2 - 2rr_0 \cos (\varphi - \varphi_0) \right]. \]
Thus
\[ v(r, \varphi; r_0, \varphi_0) = \frac{1}{4\pi} \log r_0^2 |x - x^*_0|^2 = \frac{1}{2\pi} \log r_0 - \frac{1}{2\pi} \log \frac{1}{|x - x^*_0|}. \]
We can therefore rewrite
\[ g(r, \varphi | r_0, \varphi_0) = g(x | \xi) = E(x | \xi) - E(x | \xi^*) + \frac{1}{2\pi} \log r_0. \]
(6.100)
The physical interpretation is obvious. The term \(-E(x | \xi^*)\) represents the potential of a unit negative point source located at the inverse point of \( \xi \) with respect to the unit circle; the term \((1/2\pi) \log r_0\) is a constant potential (not due to any source in the finite plane) which permits \( g \) to be 0 on \( \sigma \) rather than a nonzero constant.

**Method of Images**

According to (6.75), the Green's function \( g(x | \xi) \) is the sum of two terms, the first representing the potential \( E(x | \xi) \) due to a unit source at \( \xi \) in free space and the second being the potential \( v(x, \xi) \) due to the charge induced on
The function $v$ is harmonic for $x$ in $R$ and assumes values on $\sigma$ which just cancel $E(x|\xi)$. In some cases, it may be possible to guess the function $v(x, \xi)$. One attempts to find a relatively simple distribution of charges outside $R$ whose potential takes on the value $-E(x|\xi)$ when the observation point $x$ is on $\sigma$. This approach will work only for very simple geometries.

**EXAMPLES**

**Example 1. Green's function for the unit sphere.** We write

$$g(x|\xi) = E(x|\xi) + v(x, \xi) = \frac{1}{4\pi|x - \xi|} + v(x, \xi).$$

Our object is to represent $v(x, \xi)$ as the potential of a simple charge configuration outside the sphere. Guided by the two-dimensional case [see (6.100)], we expect that the inverse point will play a role in our search. The inverse point $\xi^*$ lies on the same radial line as $\xi$ and $|\xi^*|/|\xi| = 1$. The pivotal property enjoyed by the pair of points $\xi$ and $\xi^*$ is that

$$\frac{|s - \xi|}{|\xi|} = |s - \xi^*|,$$

whenever $s$ is on the surface of the unit sphere. The result follows from the similarity of the two triangles having vertices at $\xi, 0, s$ and $s, 0, \xi^*$, respectively.

This suggests that we should choose

$$v(x, \xi) = \frac{A}{4\pi|x - \xi^*|}.$$ 

For $g$ to vanish when $x = s$, we need

$$\frac{1}{4\pi|s - \xi|} + \frac{A}{4\pi|s - \xi^*|} = 0,$$

or, by (6.101),

$$A = -\frac{1}{|\xi|}.$$

We have therefore constructed explicitly the Green's function for the interior of the unit sphere,

$$g(x|\xi) = \frac{1}{4\pi|x - \xi|} - \frac{1}{|\xi|} \frac{1}{4\pi|x - \xi^*|},$$

where $\xi^*$ is the inverse point of $\xi$ with respect to the unit sphere.

Let us calculate $\partial g(x|\xi)/\partial n_x$. Here $\xi$ is a fixed point in the interior of the unit sphere; $\partial g(x|\xi)/\partial n_x$ is then the charge density induced on the surface of the sphere at $x$, due to a point charge at $\xi$. Using polar coordinates, we want to find $\partial g/\partial r |_{r=1}$. 

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Now
\[ |x - \xi|^2 = |\xi|^2 + r^2 - 2|\xi|r \cos \gamma, \quad |x - \xi^*|^2 = |\xi^*|^2 + r^2 - 2|\xi^*|r \cos \gamma, \]
where \( \gamma \) is the angle between the vectors \( x \) and \( \xi \). Since
\[ \frac{\partial |x - \xi|}{\partial r} = \frac{r - |\xi| \cos \gamma}{|x - \xi|}, \quad \frac{\partial |x - \xi^*|}{\partial r} = \frac{r - |\xi^*| \cos \gamma}{|x - \xi^*|}, \]
we find, from (6.102),
\[ \frac{\partial g}{\partial r} = -\frac{1}{4\pi} \left[ \frac{r - |\xi| \cos \gamma}{|x - \xi|^3} - \frac{r - |\xi^*| \cos \gamma}{|x - \xi^*|^3} \right]. \]
Therefore,
\[ \left. \frac{\partial g}{\partial r} \right|_{r=1} = -\frac{1}{4\pi} \left[ \frac{1 - |\xi| \cos \gamma}{(|\xi|^2 + 1 - 2|\xi| \cos \gamma)^{3/2}} - \frac{1 - |\xi^*| \cos \gamma}{(|\xi^*|^2 + 1 - 2|\xi^*| \cos \gamma)^{3/2}} \right]. \]
By using \( |\xi^*| = 1/|\xi| \), we have
\[ \left( \frac{\partial g}{\partial r} \right)_{r=1} = \frac{1}{4\pi} \left[ \frac{|\xi|^2 - 1}{(1 + |\xi|^2 - 2|\xi| \cos \gamma)^{3/2}} \right] = \frac{\partial g(x \mid \xi)}{\partial n_x} = \frac{\partial g(x \mid x)}{\partial n_x}, \]
and hence
\[ I(x \mid \xi) = -\frac{\partial g(x \mid \xi)}{\partial n_x} = \frac{1}{4\pi} \left[ \frac{1 - |x|^2}{[1 + |x|^2 - 2|x| \cos \gamma]^{3/2}} \right]. \tag{6.103} \]
By (6.83) we then have the solution of the Dirichlet problem in the same form as in (6.26).

**Example 2.** *The Green’s function for a half-space.* A unit source is placed at the point \( \xi \) with Cartesian coordinates \((\xi_1, \xi_2, \xi_3)\), where \( \xi_1 > 0 \). The plane \( x_1 = 0 \) is a grounded metallic conductor (that is, at zero potential). If \( R \) denotes the half-space \( x_1 > 0 \), the total potential \( g \) satisfies
\[ -\nabla^2 g = \delta(x - \xi), \quad x \text{ in } R; \quad g = 0 \text{ when } x_1 = 0. \]
To ensure uniqueness we shall also require that \( g \) vanish as \( |x| \to \infty \) in the region \( x_1 > 0 \). Writing
\[ g = \frac{1}{4\pi|x - \xi|} + v(x, \xi), \]
we must find a function \( v \) harmonic for \( x_1 > 0 \), which, on \( x_1 = 0 \), cancels the primary potential \( 1/4\pi|x - \xi| \). It obviously suffices to let \( v \) be the potential from a negative unit source placed at the image point
\[ \xi^* = (-\xi_1, \xi_2, \xi_3). \]
Then
\[ g(x \mid \xi) = \frac{1}{4\pi|x - \xi|} - \frac{1}{4\pi|x - \xi^*|}. \]  
(6.104)

Since the outward normal to \( R \) is in the \(-x_1\) direction, we have
\[ -\frac{\partial g(x \mid \xi)}{\partial n_x} = \frac{\partial g}{\partial x_1} \bigg|_{x_1=0}. \]

A simple calculation gives
\[ -\frac{\partial g(x \mid \xi)}{\partial n_x} = \frac{1}{2\pi} \frac{\xi_1}{[\xi_1^2 + (x_2 - \xi_2)^2 + (x_3 - \xi_3)^2]^{3/2}}. \]

Using the symmetry of \( g \) and relabeling variables, we have
\[ I(x \mid \xi) = -\frac{\partial g(x \mid \xi)}{\partial n_x} = \frac{1}{2\pi} \frac{x_1}{[x_1^2 + (x_2 - \xi_2)^2 + (x_3 - \xi_3)^2]^{3/2}}. \]

From (6.83) we find that
\[ u(x_1, x_2, x_3) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\xi_2 \int_{-\infty}^{\infty} d\xi_3 \frac{x_1}{[x_1^2 + (x_2 - \xi_2)^2 + (x_3 - \xi_3)^2]^{3/2}} f(\xi_2, \xi_3) \]
(6.105)
is the solution of the Dirichlet problem
\[ \nabla^2 u = 0, \quad x_1 > 0; \quad u(0, x_2, x_3) = f(x_2, x_3), \quad u \mid_{x_1=\infty} = 0. \]

In this case it is easy to verify that (6.105) complies with the prescribed boundary values on the plane \( x_1 = 0 \). We have shown in Exercise 5.2 that
\[ \lim_{x_1 \to 0^+} \frac{1}{2\pi} \frac{x_1}{[x_1^2 + (x_2 - \xi_2)^2 + (x_3 - \xi_3)^2]^{3/2}} = \delta(x_2 - \xi_2)\delta(x_3 - \xi_3). \]

**Example 3.** The Neumann function for a half-space. If the boundary condition is that the normal derivative of the potential vanishes, we have a Neumann problem. The Neumann function \( h(x \mid \xi) \) satisfies the boundary value problem
\[ -\nabla^2 h = \delta(x - \xi), \quad x \text{ in } R; \quad \frac{\partial h}{\partial x_1} = 0 \text{ when } x_1 = 0, \]
where \( R \) denotes the half-space \( x_1 > 0 \). We also require that \( h \) vanish at \( \infty \).

The method of images works equally well in this case. We choose \( v \) as the potential of a positive unit source at the image point. The sum of the primary source and \( v \) is clearly an even function of \( x_1 \), so that the \( x_1 \) derivative will vanish at \( x_1 = 0 \). Thus
\[ h = \frac{1}{4\pi|x - \xi|} + \frac{1}{4\pi|x - \xi^*|}. \]
Example 4. The two-dimensional problem for a half-space. Instead of a point source as in Example 2, we have a line source (of unit line density) parallel to the \( x_3 \) axis. The line source cuts the \( x_1 x_2 \) plane at the point \((\xi_1, \xi_2)\) where \( \xi_1 > 0 \). The potential then satisfies the two-dimensional problem

\[
-\nabla^2 g = \delta(x_1 - \xi_1)\delta(x_2 - \xi_2), \quad x_1 > 0; \quad g = 0 \quad \text{when} \ x_1 = 0.
\]

By the same line of reasoning used in Example 2, we find

\[
g(x \mid \xi) = \frac{1}{2\pi} \log \frac{1}{|x - \xi|} - \frac{1}{2\pi} \log \frac{1}{|x - \xi^*|}.
\]

The corresponding solution of the two-dimensional Dirichlet problem for the region \( x_1 > 0, -\infty < x_2 < \infty \), is

\[
u(x_1, x_2) = \frac{1}{\pi} \int_{-\infty}^{\infty} d\xi_2 \frac{x_1}{[x_1^2 + (x_2 - \xi_2)^2]} f(\xi_2).
\]

This result can also be obtained from (6.105) by taking \( f(\xi_2, \xi_3) \) independent of \( \xi_3 \).

Full Eigenfunction Expansion

We obtain an expansion for \( g \) in terms of the orthonormal eigenfunctions \( \{u_n\} \) of (6.85). Multiply (6.94) by \( \bar{u}_n \) and integrate over \( R \). This yields

\[
-\int_R \bar{u}_n \nabla^2 g \ dx = \bar{u}_n(\xi).
\]

By Green's theorem and the boundary condition

\[
\int_R \bar{u}_n \nabla^2 g \ dx = \int_R g \nabla^2 \bar{u}_n \ dx.
\]

Now \( \nabla^2 \bar{u}_n = -\lambda_n \bar{u}_n \), so that

\[
\int_R g \bar{u}_n \ dx = \langle g, u_n \rangle = \frac{\bar{u}_n(\xi)}{\lambda_n}.
\]

We have just found the Fourier coefficients of the function \( g \) in the orthonormal set \( \{u_n(x)\} \). Therefore,

\[
g(x \mid \xi) = \sum_n \frac{u_n(x) \bar{u}_n(\xi)}{\lambda_n}. \tag{6.108}
\]

Although (6.108) is of conceptual importance, it is not very useful for calculations. The eigenfunctions \( \{u_n\} \) are available to us only when we can separate variables; but in these cases the method of partial expansion (which we take up next) is simpler. This latter method yields a series which converges more rapidly and which is better suited to the solution of the Dirichlet problem.
We content ourselves with giving the explicit form of (6.108) for two special cases. For the rectangle [see (6.87) and following] we have

\[
g(x_1, x_2 | \xi_1, \xi_2) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{4}{ab} \times \frac{\sin (m\pi x_1/a) \sin (m\pi x_2/b) \sin (n\pi \xi_1/a) \sin (n\pi \xi_2/b)}{(m^2\pi^2/a^2) + (n^2\pi^2/b^2)},
\]

(6.109)

and, for the circle [see (6.88) and following],

\[
g(r, \varphi | r_0, \varphi_0) = \frac{1}{\pi} \sum_{n=\infty}^{\infty} \sum_{k=1}^{\infty} e^{in\varphi} e^{-in\varphi_0} J_n(\beta_k^{(n)} r) J_n(\beta_k^{(n)} r_0) \left[ J'_n(\beta_k^{(n)}) \right]^2 \left[ \beta_k^{(n)} \right]^2.
\]

(6.110)

**Partial Eigenfunction Expansion**

The method about to be described can be used not only to find the Green's function but also to solve directly the Dirichlet problem for Poisson's equation. To fix ideas, we shall consider a two-dimensional problem, although this restriction is not essential. Let \( R \) be a region which is defined in the curvilinear coordinate system \((s, t)\) by \( s_1 < s < s_2, \ t_1 < t < t_2 \). In general then, \( R \) is a curvilinear quadrilateral whose sides correspond to a constant value of \( s \) or \( t \). In some cases the quadrilateral degenerates into a simpler region; for instance, in polar coordinates, \( R \) could be the wedge \( 0 < r < \infty, \ 0 < \varphi < \alpha \), or the full circle \( 0 < r < 1, \ -\pi < \varphi < \pi \). These degenerate cases occur when one of the sides of \( R \) is infinite, or when the Jacobian of the transformation from Cartesian coordinates vanish at some point in \( R \), or when the transformation from Cartesian coordinates is not continuously differentiable in the region \( R \). Although the method which we shall outline can be applied in degenerate cases (as will be seen in some of the examples), we shall restrict ourselves for the present to a genuine finite quadrilateral.

The crucial property needed for the success of the procedure is that \( \nabla^2 \) can be separated in the \( s, t \) coordinate system. To be precise, this means that the homogeneous equation \( \nabla^2 u = 0 \) can be written

\[
\frac{1}{a_1(s)} L_1 u + \frac{1}{a_2(t)} L_2 u = 0,
\]

where \( L_1 \) is a differential operator which depends only on the variable \( s \), and \( L_2 \) only on \( t \). Typically these operators are formally self-adjoint and of the second order.

If we look for separable solutions of the form

\[
u = S(s)T(t),
\]

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we are led to the one-dimensional eigenvalue problems

\[ L_1 S = \lambda a_1(s)S, \quad s_1 < s < s_2; \quad S(s_1) = S(s_2) = 0; \]
\[ L_2 T = \mu a_2(t)T, \quad t_1 < t < t_2; \quad T(t_1) = T(t_2) = 0. \]

Each eigenvalue problem gives rise to a complete orthonormal set of eigenfunctions (possibly with a weight function).

The method for finding the Green’s function \( g \) for \( R \) then consists of expanding \( g \) in terms of the eigenfunctions of either of the one-dimensional problems. We thus find two different representations for \( g \), each of which is useful in different contexts.

**Examples**

**Example 1.** Green’s function for the rectangle. Let \( R \) be the two-dimensional region \( 0 < x < a, 0 < y < b \), where \( x \) and \( y \) are the usual Cartesian coordinates. We consider the problem

\[ -\frac{\partial^2 g}{\partial x^2} - \frac{\partial^2 g}{\partial y^2} = \delta(x - \xi)\delta(y - \eta), \quad 0 < x, \xi < a, \quad 0 < y, \eta < b; \]
\[ g \big|_{x=0} = g \big|_{x=a} = 0, \quad 0 < y < b; \]
\[ g \big|_{y=0} = g \big|_{y=b} = 0, \quad 0 < x < a, \quad (6.111) \]

The one-dimensional problems obtained by separating \( \nabla^2 u = 0 \) in Cartesian coordinates with the above boundary conditions yield the complete orthogonal sets \( \{ \sin(m\pi x/a) \} \) and \( \{ \sin(n\pi y/b) \} \). Our procedure will consist of expanding \( g \) in one of these sets. The coefficients of the expansion will satisfy an inhomogeneous equation in the other independent variable. This inhomogeneous equation is actually itself a Green’s function problem, but now one-dimensional. The problem can then be solved explicitly by the methods of Chapter 1.

Let us carry out the analysis. For each fixed \( y \) we can expand \( g \) in a series in \( \sin(m\pi x/a) \) and the coefficients will depend on \( y, \xi, \) and \( \eta \). To simplify the notation we suppress the dependence on \( \xi \) and \( \eta \). Thus

\[ g = \sum_{m=1}^{\infty} g_m(y) \sin \frac{m\pi x}{a}, \]

where

\[ g_m(y) = \frac{2}{a} \int_0^a g \sin \frac{m\pi x}{a} \, dx. \quad (6.112) \]

We shall calculate the coefficients \( g_m \) by multiplying the differential equation in (6.111) by \( (2/a) \sin(m\pi x/a) \) and integrating from 0 to \( a \). This gives

\[ -\frac{2}{a} \int_0^a \sin \frac{m\pi x}{a} \frac{\partial^2 g}{\partial x^2} \, dx - \frac{2}{a} \int_0^a \sin \frac{m\pi x}{a} \frac{\partial^2 g}{\partial y^2} \, dx = \frac{2}{a} \sin \frac{m\pi \xi}{a} \delta(y - \eta). \]
In the second integral we may interchange the orders of differentiation and integration. The first integral is integrated by parts twice; using the boundary conditions on \( g \) and \( \sin \left( \frac{m\pi x}{a} \right) \), we find

\[
\frac{d^2 g_m}{dy^2} + \frac{m^2 \pi^2}{a^2} g_m = \frac{2}{a} \sin \frac{m\pi \xi}{a} \delta(y - \eta).
\]

Since \( g \) vanishes at \( y = 0 \) and \( y = b \), it follows from (6.112) that \( g_m \) also does. We therefore have the associated boundary conditions

\[
g_m(0) = g_m(b) = 0.
\]

The one-dimensional Green's function \( g_m \) can be explicitly calculated. For \( y \neq \eta \), we must have a solution of the homogeneous equation. Taking into account the boundary conditions and the continuity of \( g_m \) at \( y = \eta \), we obtain

\[
g_m = C \sinh \frac{m\pi}{a} (y_\prec) \sinh \frac{m\pi}{a} (b - y_\succ),
\]

where

\[
y_\prec = \min (y, \eta), \quad y_\succ = \max (y, \eta).
\]

The jump condition on the derivative of \( g_m \) at \( y = \eta \) is

\[
g_m^\prime(\eta^+) - g_m^\prime(\eta^-) = -\frac{2}{a} \sin \frac{m\pi \xi}{a},
\]

from which we find, after some simplifications,

\[
g_m(y) = \frac{2}{m\pi} \frac{\sin \left( \frac{m\pi \xi y}{a} \right)}{\sinh \left( \frac{m\pi \xi}{a} \right)} \sinh \frac{m\pi}{a} (y_\prec) \sinh \frac{m\pi}{a} (b - y_\succ).
\]

The Green's function \( g \) is then given by

\[
g = \sum_{m=1}^{\infty} g_m(y) \sin \frac{m\pi x}{a}. \tag{6.113}
\]

The expression (6.113) is particularly useful for calculating \( g \) at points whose \( y \) coordinate differs from \( \eta \). For \( m \) large we can replace the hyperbolic functions by positive exponentials. We then find that the series is dominated by one whose ratio of successive terms is approximately

\[
e^{-|y-\eta|/(\pi/a)}.
\]

It is clear that we have essentially a geometric series with ratio smaller than 1; the farther \( y \) is from \( \eta \), the smaller the ratio, and the faster the series converges.

If we had started instead with an expansion in the \( y \) eigenfunctions \( \{\sin (n\pi y/b)\} \), we would have found

\[
g = \sum_{n=1}^{\infty} g_n(x) \sin \frac{n\pi y}{b}, \tag{6.114}
\]
where
\[ g_n(x) = \frac{2}{n\pi \sinh(n\pi a/b)} \sinh \frac{n\pi}{b} (x_<) \sinh \frac{n\pi}{b} (a - x_>) , \]
\[ x_< = \min (x, \xi) , \quad x_> = \max (x, \xi) . \]

The expression (6.114) converges rapidly if \(|x - \xi|\) is not too small.

It should also be pointed out that in (6.113) or (6.114) we can carry out a further expansion. For instance in (6.113) we can expand \(g_m(y)\) in the set \{\sin (n\pi y/b)\}. This is most easily done by backtracking to the differential equation for \(g_m\). Writing
\[ g_m(y) = \sum_{n=1}^{\infty} g_{m,n} \sin \frac{n\pi y}{b} ; \quad g_{m,n} = \frac{2}{b} \int_0^b g_m(y) \sin \frac{n\pi y}{b} \, dy , \]
we obtain after multiplying the differential equation by \((2/b) \sin (n\pi y/b)\) and integrating from 0 to \(b\),
\[ \left( \frac{n^2 \pi^2}{b^2} + \frac{m^2 \pi^2}{a^2} \right) g_{m,n} = \frac{4}{ab} \sin \frac{m\pi \xi}{a} \sin \frac{n\pi \eta}{b} . \]
Thus
\[ g = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{4}{ab} \sin \frac{m\pi \xi}{a} \sin \frac{n\pi \eta}{b} \sin \frac{(m\pi x/a) \sin (n\pi \eta/b) \sin (m\pi x/a) \sin (n\pi y/b)}{(m^2 \pi^2/a^2) + (n^2 \pi^2/b^2)} . \quad (6.115) \]

In this case (which is not typical), this double expansion reduces to the expansion (6.109) in the full set of eigenfunctions of the two-dimensional eigenvalue problem for the negative Laplacian.

Let us now see how (6.113) or (6.114) can be used to solve the Dirichlet problem for the rectangle. This Dirichlet problem is the superposition of four part-problems in each of which only one side of the rectangle carries non-vanishing boundary values. A typical part-problem is then
\[ \nabla^2 u = 0 , \quad 0 < x < a , \quad 0 < y < b ; \]
\[ u(x, b) = f(x) , \quad 0 < x < a ; \]
\[ u(x, 0) = 0 , \quad 0 < x < a ; \quad (6.116) \]
\[ u(0, y) = u(a, y) = 0 , \quad 0 < y < b . \]

According to (6.83) the solution is
\[ u(x, y) = \int_0^a I(x, y | \xi, b) f(\xi) d\xi , \]
where
\[ I = - \left( \frac{\partial g}{\partial \eta} \right)_{\eta=b} . \]
To calculate $I$, we first need $g$ for $\eta > y$. From (6.113)

$$g(x, y \mid \xi, \eta) = \sum_{m=1}^{\infty} \frac{2}{m \pi} \sin \frac{m \pi x}{a} \sin \frac{m \pi \xi}{a} \times \sinh \frac{m \pi y}{a} \sinh \frac{m \pi}{a} (b - \eta) \frac{1}{\sinh (m \pi b/a)}.$$ 

Thus

$$- \frac{\partial g}{\partial \eta} \bigg|_{\eta=b} = \frac{2}{a} \sum_{m=1}^{\infty} \sin \frac{(m \pi \xi/a)}{\sinh (m \pi b/a)} \sinh \frac{m \pi y}{a} \sin \frac{m \pi x}{a},$$

$$u(x, y) = \frac{2}{a} \sum_{m=1}^{\infty} \sin \frac{m \pi x}{a} \sinh \frac{(m \pi y/a)}{\sinh (m \pi b/a)} \int_{0}^{a} f(\xi) \sin \frac{m \pi \xi}{a} \, d\xi. \quad (6.117)$$

The solution (6.117) of (6.116) is more easily obtained by expanding $u$ directly in a series in the set $\{ \sin (m \pi x/a) \}$ without going through the intermediary of the Green's function (as was done in Exercise 6.13).

The series (6.117) converges rapidly for all $y$ which are not too close to $b$. On a naive level we can see that the boundary values are attained at $y = b$; in fact, the series then reduces to the sine expansion of $f(x)$. Of course, what one should really prove is that

$$\lim_{y \to b^-} u(x, y) = f(x),$$

or, even better,

$$\lim_{y \to b^-} \int_{0}^{a} |u(x, y) - f(x)|^2 \, dx = 0.$$ 

This last property was proved in Exercise 6.13.

One could solve (6.116) by using the representations (6.114) or (6.115), but the resulting series would converge too slowly to be of practical value. On the other hand, when solving a part-problem in which the nonvanishing boundary values are on $x = 0$ or $x = a$, the representation (6.114) should be used instead of (6.113).

To illustrate the rapid convergence of (6.117), we consider the problem of steady heat conduction in a square plate of side $a$, when $f(x) = 1$. Then (6.117) reduces to

$$u(x, y) = \frac{4}{\pi} \sum_{m \text{ odd}} \frac{\sin (m \pi x/a) \sinh (m \pi y/a)}{m \sinh m \pi}.$$ 

At the center of the plate ($x = a/2, y = a/2$), we find the temperature

$$u_0 = \frac{4}{\pi} \left[ \frac{\sinh (\pi/2)}{\sinh \pi} - \frac{1}{3} \frac{\sinh (3\pi/2)}{\sinh 3\pi} + \cdots \right].$$
The second term is already very small compared to the first; using the favorite weapon of applied mathematicians—the one-term approximation to series—we obtain the estimate

\[ u_0 \sim 0.253. \]

The exact value of \( u_0 \) is \( \frac{1}{4} \), as was shown by symmetry arguments in Exercise 6.4.

**Example 2.** Green’s function for the unit circle. We place the source at \( \phi = 0, r = r_0 \). The Green’s function will then depend on \( r, r_0 \), and \( \phi \), but we suppress the dependence on \( r_0 \). After expressing the delta function in polar coordinates [see (5.23)], we have the boundary value problem

\[
- \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial g}{\partial r} \right) - \frac{1}{r^2} \frac{\partial^2 g}{\partial \phi^2} = \frac{\delta(r - r_0)\delta(\phi)}{r}, \quad r < 1; \tag{6.118}
\]

\[ g \mid_{r=1} = 0. \]

If we separate \( \nabla^2 u = 0 \) in polar coordinates for a full \( 2\pi \) range in \( \phi \), we find that the one-dimensional problem in \( \phi \) yields the complete orthogonal set \( \{e^{i n \phi}\} \). We therefore expand \( g \) in this set:

\[ g = \sum_{n=-\infty}^{\infty} g_n(r)e^{in\phi}, \quad g_n(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} ge^{-in\phi} \, d\phi. \]

We multiply the differential equation in (6.118) by \((1/2\pi)e^{-in\phi}\) and integrate from \(-\pi\) to \(\pi\). The second term is integrated by parts twice; using the fact that

\[ g \mid_{\phi=\pi} = g \mid_{\phi=-\pi}; \quad \frac{\partial g}{\partial \phi} \mid_{\phi=\pi} = \frac{\partial g}{\partial \phi} \mid_{\phi=-\pi}, \]

we find

\[ -\frac{d}{dr} \left( r \frac{d g_n}{dr} \right) + \frac{n^2}{r} g_n = \frac{1}{2\pi} \delta(r - r_0), \quad n = \ldots, -2, -1, 0, 1, 2, \ldots. \]

The solution of the homogeneous equation for \( n \neq 0 \) is \( Ar^n + Br^{-n} \). Since \( g_n \) must be bounded at \( r = 0 \), we retain the solution \( r^{|n|} \) for \( r < r_0 \). For \( r > r_0 \) we use the solution \( r^{|n|} - r^{-|n|} \), which obeys the boundary condition \( g_n(1) = 0 \). Thus, for \( n \neq 0 \),

\[ g_n(r) = \begin{cases} Ar^{|n|}, & r < r_0; \\ B(r^{|n|} - r^{-|n|}), & r > r_0. \end{cases} \]

Similarly,

\[ g_0(r) = \begin{cases} A, & r < r_0; \\ B \log r, & r > r_0. \end{cases} \]
Now $g_n$ is continuous at $r = r_0$, and the jump condition on $g'_n$ is
\[ g'_n(r_0^+) - g'_n(r_0^-) = -\frac{1}{2\pi r_0}. \]
These conditions enable us to calculate $A$ and $B$. The result is
\[ g_n(r) = -\frac{1}{4\pi|n|}(r_{<}^{n}|r_{>}^{n}| - r_{>}^{-n}|), \quad n \neq 0; \]
\[ g_0(r) = -\frac{1}{2\pi} \log r_>. \]
Here
\[ r_> = \max (r, r_0), \quad r_< = \min (r, r_0). \]
We then conclude that
\[ g = -\frac{1}{2\pi} \log r_> - \frac{1}{4\pi} \sum_{n \neq 0}^{\infty} \frac{e^{\text{imp}}}{|n|} (r_{<}^{n}|r_{>}^{n}| - r_{>}^{-n}|), \]
or
\[ g = -\frac{1}{2\pi} \log r_> - \frac{1}{2\pi} \sum_{n=1}^{\infty} \cos \frac{n\varphi}{n} \frac{n}{|n|} (r_{<}^{n}|r_{>}^{n}| - r_{>}^{-n}|). \]
If the source is at $\varphi = \varphi_0$ instead of at $\varphi = 0$, we merely rotate the solution by the amount $\varphi_0$. Thus
\[ g(r, \varphi | r_0, \varphi_0) = -\frac{1}{2\pi} \log r_> - \frac{1}{2\pi} \sum_{n=1}^{\infty} \cos \frac{n(\varphi - \varphi_0)}{n} (r_{<}^{n}|r_{>}^{n}| - r_{>}^{-n}|). \]
(6.119)
By an easy calculation using (6.22), we can reduce this expression to (6.99).
To solve the Dirichlet problem for the unit circle, we need to calculate
\[ I = -\left(\frac{\partial g}{\partial r_0}\right)_{r_0=1} \]
For $r_0 > r$, we have
\[ g = -\frac{1}{2\pi} \log r_0 - \frac{1}{2\pi} \sum_{n=1}^{\infty} \cos \frac{n(\varphi - \varphi_0)}{n} r^n(r_0^n - r_0^{-n}), \]
\[ -\left(\frac{\partial g}{\partial r_0}\right)_{r_0=1} = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{n=1}^{\infty} r^n \cos n(\varphi - \varphi_0). \]
Thus the solution of the Dirichlet problem is
\[ u(r, \varphi) = \int_{-\pi}^{\pi} \left[ \frac{1}{2\pi} + \frac{1}{\pi} \sum_{n=1}^{\infty} r^n \cos n(\varphi - \varphi_0) \right] f(\varphi_0) d\varphi_0, \]
which is easily seen to be equivalent to (6.9). In Exercise 6.39 we show how the Green's function of (6.118) can be expanded in a set of radial eigenfunctions.

**Example 3.** Green's function for a strip. We consider a line source parallel to the z axis and cutting the xy plane at the point \( x = 0, \ y = \eta \), where \( 0 < \eta < a \). Infinite plates at 0 potential coincide with the planes \( y = 0 \) and \( y = a \). Our problem is clearly two-dimensional; the potential \( g(x, y \mid 0, \eta) \) satisfies

\[
-\nabla^2 g = \delta(x)\delta(y - \eta); \quad 0 < y, \eta < a, \quad -\infty < x < \infty;
\]

\[
g\big|_{y=0} = g\big|_{y=a} = 0.
\]

If we separate the homogeneous equation, we obtain the two one-dimensional eigenvalue problems

\[
-\ Y'' = \lambda Y, \quad 0 < y < a; \quad Y(0) = Y(a) = 0;
\]

\[
-\ X'' = \mu X, \quad -\infty < x < \infty.
\]

The first problem leads to the eigenvalues \( \lambda_n = n^2\pi^2/\alpha^2 \) with the eigenfunctions \( \sin(n\pi y/a) \). The second problem is a singular Sturm-Liouville problem, studied in Chapter 4. For our purposes it suffices to know that there is a natural connection between this singular problem and the Fourier integral. Speaking loosely, every real positive number \( \mu = \alpha^2 \) is in the spectrum and we can associate with \( \alpha^2 \) the pseudo eigenfunction \( e^{i\alpha x}, \ -\infty < \alpha < \infty \). In any event we are led to using a Fourier transform on the x coordinate. Suppressing temporarily the dependence on \( \eta \), we define the Fourier transform of \( g(x, y) \) as

\[
g^\wedge(\alpha, y) = \int_{-\infty}^{\infty} g(x, y) e^{i\alpha x} \, dx.
\]

We have the inversion formula

\[
g(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g^\wedge(\alpha, y) e^{-i\alpha x} \, dx.
\]

Multiply the differential equation in (6.120) by \( e^{iax} \) and integrate from \( x = -\infty \) to \( x = +\infty \). We obtain

\[
-\int_{-\infty}^{\infty} e^{iax} \frac{\partial^2 g}{\partial y^2} \, dx - \int_{-\infty}^{\infty} e^{iax} \frac{\partial^2 g}{\partial x^2} \, dx = \delta(y - y_0).
\]

In the first integral we interchange the order of integration and differentiation. The second integral is integrated by parts twice and the integrated terms at \( \pm \infty \) are dropped (it is not necessary to argue persuasively that this step is legitimate; we merely check afterward that we have actually found the solution of the original boundary value problem). We have

\[
-\frac{d^2 g^\wedge}{dy^2} + \alpha^2 g^\wedge = \delta(y - \eta),
\]
and, since $g$ vanishes at $y = 0$ and at $y = a$, it follows from (6.121) that $g^\wedge$ also vanishes for these values of $y$.

By a simple calculation, we find

$$g^\wedge = \frac{1}{\alpha \sinh \alpha a} \sinh \alpha (y_\prec) \sinh \alpha (a - y_\succ),$$

where $y_\prec = \min (y, \eta)$, $y_\succ = \max (y, \eta)$. Thus, from (6.122), we have

$$g = \frac{1}{2\pi} \int_{-\infty}^{\infty} g^\wedge e^{-i\alpha x} \, d\alpha,$$

and if the source is placed at $x = \xi$ instead of at $x = 0$,

$$g(x, y | \xi, \eta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g^\wedge e^{-i\alpha (x - \xi)} \, d\alpha$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\alpha}{\alpha \sinh \alpha a} \sinh \alpha (y_\prec) \sinh \alpha (a - y_\succ) e^{-i\alpha (x - \xi)}. \quad (6.123)$$

A typical Dirichlet problem for the strip is

$$\nabla^2 u = 0, \quad 0 < y < a, \quad -\infty < x < \infty; \quad u(x, 0) = 0, \quad u(x, a) = f(x). \quad (6.124)$$

To use (6.83) to solve this Dirichlet problem, we need

$$I = -\left( \frac{\partial g}{\partial \eta} \right)_{\eta = a}. $$

Differentiating (6.123) for $\eta > y$ and setting $\eta = a$, we find

$$I = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sinh \alpha y}{\sinh \alpha a} e^{-i\alpha (x - \xi)} \, d\alpha. \quad (6.125)$$

This integral is evaluated in closed form in Exercise 6.36. In any event the solution of (6.124) is

$$u(x, y) = \int_{-\infty}^{\infty} I(x, y, \xi) f(\xi) \, d\xi. \quad (6.126)$$

The expression (6.126) can, of course, be derived more easily by applying a Fourier transform directly to (6.124).

One can also calculate the solution of (6.120) by using an expansion in the $y$ eigenfunctions $\{ \sin (n\pi y/a) \}$. We write

$$g = \sum_{n=1}^{\infty} g_n(x) \sin \frac{n\pi y}{a}, \quad g_n = \frac{2}{a} \int_{0}^{a} g \sin \frac{n\pi y}{a} \, dy.$$

Multiply the differential equation in (6.120) by $(2/a) \sin (n\pi y/a)$ and integrate
from \( y = 0 \) to \( y = a \) to obtain

\[
-g''_n(x) + \frac{n^2 \pi^2}{a^2} g_n = \frac{2}{a} \sin \frac{n \pi y}{a} \delta(x), \quad -\infty < x < \infty.
\]

Requiring that \( g_n(x) \) vanish at \( |x| = \pm \infty \), we find

\[
g_n(x) = \frac{e^{-(n\pi/a)|x|}}{n\pi} \sin \frac{n \pi y}{a}.
\]

\[
g = \sum_{n=1}^{\infty} \frac{e^{-(n\pi/a)|x|}}{n\pi} \sin \frac{n \pi y}{a} \sin \frac{n \pi y}{a}.
\]

If the source is placed at \( x = \xi \) instead of at \( x = 0 \), we have

\[
g(x, y | \xi, \eta) = \sum_{n=1}^{\infty} \frac{e^{-(n\pi/a)|x-\xi|}}{n\pi} \sin \frac{n \pi y}{a} \sin \frac{n \pi y}{a}, \quad (6.127)
\]

which should be compared with (6.123).

One can show, by using the Poisson summation formula (1.23a) derived in Chapter 1, that (6.127) is indeed equal to (6.123). We calculate \( I = -\frac{\partial g}{\partial \eta}_{\eta=a} \) from (6.127). This gives

\[
I = \frac{1}{a} \sum_{n=1}^{\infty} \frac{(-1)^n}{n\pi} \sin \frac{n \pi y}{a} e^{-(n\pi/a)|x-\xi|}. \quad (6.128)
\]

In the above form, \( I \) is not particularly useful for solving the Dirichlet problem (6.124). It is clear that after inserting \( I \) in (6.126) we will have a series expansion in the set \( \{ \sin (n \pi y/a) \} \). Since each term vanishes at \( y = a \), it will be difficult to verify the boundary condition

\[
\lim_{y \to a^-} u(x, y) = f(x).
\]

In fact, the series for \( u \) does not converge uniformly in \( 0 < y < a \) and calculations with the series will be cumbersome. Instead we rewrite (6.128) in a more tractable form by noting that we have essentially a geometric series. To be precise,

\[
I = -\frac{1}{a} \Im \left( \sum_{n=1}^{\infty} \theta^n \right),
\]

where

\[
\theta = -e^{iy/a} e^{-|x-\xi|\pi/a}.
\]

An easy calculation then shows that

\[
I = \frac{\sin (\pi y/a) e^{-|x-\xi|\pi/a}}{a \left( 1 + 2e^{-|x-\xi|\pi/a} \cos (\pi y/a) + e^{-2|x-\xi|\pi/a} \right)}
\]
or

\[
I = \frac{1}{2a} \frac{\sin (\pi y/a)}{\cos (\pi y/a) + \cosh ((x - \xi)\pi/a)}. \tag{6.129}
\]

This expression can then be substituted in (6.126) to obtain the solution of the Dirichlet problem (6.124) in its simplest form.

**Complex Variable Method for Two-Dimensional Problems**

The remarkable efficacy of the theory of functions of a complex variable in dealing with Laplace's equation in two dimensions stems from the property that the real and imaginary parts of an analytic function of a complex variable are harmonic functions. We shall not give a comprehensive account of the use of this theory; instead we will show only how the mapping function enables us to determine the Green’s function for a plane region.

Our principal tool is the Riemann mapping theorem, which we state without proof. Let \( R \) be a plane, simply connected region with boundary \( C \) and let \( z_0 = x_0 + iy_0 \) be a fixed point in the interior of \( R \). Then there exists a function \( w = f(z) \), analytic at all interior points of \( R \), which maps \( R \) in a one-to-one manner onto the unit circle in the \( w \) plane in such a way that \( C \) is mapped onto the circumference of the unit circle and \( z_0 \) is mapped into the origin \( w = 0 \). Moreover, the mapping is conformal at every interior point of \( R \).

**Remarks.**

1. A mapping is said to be conformal at a point if the angle between any two line elements through the point is preserved under the mapping. One can show easily that this implies \( f''(z) \neq 0 \).
2. Since the mapping function \( f \) depends on which point \( z_0 \) is mapped into \( w = 0 \), it will be preferable to denote it by \( f(z, z_0) \).
3. If \( f(z, z_0) \) is a mapping function, so is \( e^{i\alpha}f(z, z_0) \) for any real \( \alpha \). In fact, the multiplicative factor \( e^{i\alpha} \) merely rotates the unit circle in the \( w \) plane by an angle \( \alpha \).
4. Except for the factor \( e^{i\alpha} \), \( f(z, z_0) \) is uniquely determined.

In terms of \( f(z, z_0) \), we claim that the Green’s function for \( R \) can be expressed as

\[
g(x, y \mid x_0, y_0) = -\frac{1}{2\pi} \log |f(z, z_0)| = -\frac{1}{2\pi} \Re \log f. \tag{6.130}
\]

We observe that \( e^{i\alpha}f \) yields the same \( g \), since \( |e^{i\alpha}| = 1 \). Let us now verify that (6.130) meets all the requirements on the Green’s function. If \( z \) is on \( C \), then \( f \) is on the circumference of the unit circle so that \( |f| = 1 \) and \( g = 0 \). When \( z \neq z_0 \), \( \log f \) is analytic and its real part is harmonic; thus \( \nabla^2 g = 0 \) at \( z \neq z_0 \).

It remains to show that the singularity of \( g \) at \( z_0 \) corresponds to a unit source.
This will be the case if we can prove that
\[ \lim_{\epsilon \to 0} \int_{C_{\epsilon}} \frac{\partial g}{\partial n} \, dl = -1, \]
where \( C_{\epsilon} \) is a circle of radius \( \epsilon \) with center at \( z_0 \). For \( z \) near \( z_0 \), \( f = f_0 + (df/dz)_0(z - z_0) + \cdots \), where the subscript 0 on a function indicates it is to be evaluated at \( z = z_0 \). Since \( f \) maps \( z_0 \) into \( w = 0, f_0 = 0 \), and, since the mapping is conformal, \( (df/dz)_0 \neq 0 \). Therefore, for \( z \) near \( z_0 \),
\[ -\frac{\partial g}{\partial n} = \frac{1}{2\pi} \frac{\partial}{\partial n} \log |f| = \frac{1}{2\pi |f|} \frac{\partial |f|}{\partial n} \sim \frac{1}{2\pi \epsilon}. \]
From this estimate for \( -\frac{\partial g}{\partial n} \), the desired integral relation follows.

The problem of finding the Green’s function \( g \) has therefore been reduced to determining the mapping function \( f(z, z_0) \). Admittedly we have merely transferred the difficulties to another branch of mathematics, but we can now avail ourselves of the elegant and far-reaching results of a well-developed theory.

We now construct the Green’s function for a circle of radius 1 by use of the mapping function which transforms the interior of the unit circle into itself and takes the fixed but arbitrary point \( z_0 \) into the origin. In the theory of functions, one is led to consider for this purpose the linear fractional transformation
\[ w = \frac{A + Bz}{C + Dz}. \]
It can be shown that such a transformation maps circles into circles (straight lines are degenerate circles), and inverse points into inverse points. Now \( z_0 \) and \( 1/z_0 \) are inverse points with respect to the unit circle; since \( w(z_0) = 0 \), we must have \( w(1/z_0) = \infty \). These two conditions simplify \( w \),
\[ w = f(z, z_0) = E \frac{z - z_0}{1 - z z_0}. \]
The constant \( E \) is determined by the requirement that points on the circumference of the unit circle in the \( z \) plane are mapped into points on the circumference of the unit circle in the \( w \) plane; that is, when \( zz = 1 \), we have \( w \bar{w} = 1 \). Now
\[ w \bar{w} = EE \left( \frac{zz_0 - z_0 \bar{z} - z \bar{z}_0 + z_0 \bar{z}_0}{1 - \bar{z}z_0 - z \bar{z}_0 - \bar{z}z_0 \bar{z}_0} \right); \]
when \( zz = 1 \), we will have \( w \bar{w} = 1 \) if and only if \( EE = 1 \). Therefore, \( E = e^{i\alpha} \) with \( \alpha \) real. The mapping function is
\[ w = e^{i\alpha} \frac{z - z_0}{1 - z z_0}, \]
and the Green's function for the unit circle is

\[ g(x, y | x_0, y_0) = -\frac{1}{2\pi} \log \left| \frac{z - z_0}{1 - z\overline{z}_0} \right|, \]

which is easily shown to be equal to (6.100).

We can now simplify somewhat the problem of finding \( f(z, z_0) \) for an arbitrary region \( R \). There are two difficulties related to determining \( f \), the principal one is to find any function whatever which maps \( R \) onto the unit circle and the secondary one is to impose the requirement that the specified point \( z_0 \) be mapped into the origin. We are now in a position to remove the secondary difficulty. Suppose we can find a transformation \( s(z) \) which maps \( R \) onto the unit circle conformally, but maps \( z_0 \) into the point \( s_0 = s(z_0) \), where \( s_0 \) is not necessarily 0. If we then apply the mapping \( w = e^{i\pi/2}[(s - s_0)/(1 - s\overline{s}_0)] \) we will bring \( s_0 \) into the origin while mapping the unit circle conformally onto itself. Therefore the mapping

\[ f(z, z_0) = e^{i\pi/2} \frac{s(z) - s(z_0)}{1 - s(z)\overline{s}(z_0)} \]

maps \( R \) onto the unit circle and takes \( z_0 \) into the origin; hence the Green's function for \( R \) is

\[ g(x, y | x_0, y_0) = -\frac{1}{2\pi} \log \left| \frac{s(z) - s(z_0)}{1 - s(z)\overline{s}(z_0)} \right|. \]  \hspace{1cm} (6.131)

As an example, let us find the Green's function for a circle of radius \( a \). The mapping \( z/a \) takes this circle into the unit circle, and therefore

\[ g(x, y | x_0, y_0) = -\frac{1}{2\pi} \log \left| \frac{z/a - z_0/a}{1 - z\overline{z}_0/a^2} \right|. \]

**Exercises**

6.35 Use the method of images to find the Green's function for a strip (6.120). Positive image sources have to be placed at \( y = \eta + 2na \) and negative sources at \( y = -\eta + 2na \), where \( n \) ranges through all integers. Thus

\[ g(x, y | 0, \eta) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left[ \log \frac{1}{[x^2 + (y - \eta - 2na)^2]^{1/2}} \right. \\
- \left. \log \frac{1}{[x^2 + (y + \eta - 2na)^2]^{1/2}} \right] \]

Show that this series reduces to (6.127) with \( \xi = 0 \).

6.36 By the calculus residues show that, for \( 0 < y < a \),

\[ \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sin \alpha y}{\sin \pi x} e^{-i\alpha (x - \xi)} d\alpha = \frac{1}{2a} \frac{\sin (\pi y/a)}{\cos (\pi y/a) + \cosh (|x - \xi|\pi/a)}. \]
This then establishes that the Fourier transform approach (6.125) and the series expansion approach yield the same answer for the Dirichlet problem (6.124).

6.37 Consider again the Green's function for the strip (6.120). Use the method of integral equations (6.97) to find a pair of simultaneous integral equations to determine the charge densities on the planes \( y = 0 \) and \( y = a \). Solve these equations by Fourier transforms. In particular, obtain (6.125).

6.38 Find, by images, the Green's function for the two-dimensional region

\[
a < r < \infty, \quad 0 < \varphi < \frac{\pi}{2}.
\]

Does the method work for the complementary region?

6.39 Consider again the Green's function for the unit circle [see (6.118)]. We want to write \( g \) as an expansion in radial eigenfunctions. When we separate Laplace's equation in polar coordinates we find the radial equation

\[
-(rR')' = \frac{\lambda R}{r}, \quad 0 < r < 1; \quad R(1) = 0.
\]

This problem leads to a continuous spectrum (see Exercise 4.28). The natural transform associated with the problem is the Mellin sine transform defined in (4.135),

\[
\hat{F}_s(v) = \int_0^1 \frac{1}{r} [\sin (v \log r)] f(r) dr,
\]

with the inversion (4.136),

\[
f(r) = \frac{2}{\pi} \int_0^\infty [\sin (v \log r)] \hat{F}_s(v) dv.
\]

Before applying a Mellin sine transform to (6.118), a preliminary change of dependent variable will avoid convergence difficulties at \( r = 0 \). Let

\[
g = g_0 + h,
\]

where \( g_0 \) is the value of \( g \) at the origin. Apply the Mellin sine transform to the equation for \( h \). Show that

\[
\hat{h}_s = \frac{g_0}{v} + \hat{K}_s,
\]
where
\[ -\nu^2 \ddot{\dot{K}}_s + \frac{d^2 \ddot{K}_s}{d\varphi^2} = -\delta(\varphi) \sin(\nu \log r_0). \]

It is convenient here to let the polar coordinate $\varphi$ have the range $(0, 2\pi)$. Show that
\[ \ddot{K}_s = \frac{\sin(\nu \log r_0)}{2\nu \sinh \nu \pi} \cosh\nu(\varphi - \pi), \]

\[ h = \frac{2}{\pi} \int_0^\infty \left[ \frac{g_0}{\nu} + \ddot{K}_s \right] \sin(\nu \log r) \, dv. \]

Since the first integral is $-g_0$, we find
\[ g(r, \theta | r_0, 0) = \frac{1}{\pi} \int_0^\infty \frac{\sin(\nu \log r) \sin(\nu \log r_0)}{\nu \sinh \nu \pi} \cosh\nu(\varphi - \pi) \, dv. \quad (6.132) \]

Show that (6.132) is actually convergent and that $g_0 = -(1/2\pi) \log r_0$, which agrees with the value obtained from (6.119).

Returning to the differential equation for $\ddot{K}_s$, expand in the set \{cos $n\varphi$\} and show that
\[ g = \frac{1}{\pi^2} \int_0^\infty \frac{\sin(\nu \log r) \sin(\nu \log r_0)}{\nu^2} \, dv \]
\[ + \frac{2}{\pi^2} \sum_{n=1}^{\infty} \cos n\varphi \int_0^\infty \frac{\sin(\nu \log r) \sin(\nu \log r_0)}{\nu^2 + n^2} \, dv. \]

By contour integration show that this expression for $g$ reduces to (6.119).

6.40 Let $R$ denote the two-dimensional region given in polar coordinates by
\[ a < r < \infty; \quad 0 < \varphi < \frac{\pi}{2}. \]

A source is placed at $r_0, \varphi_0$ within $R$. The boundary conditions are that the potential vanishes on the circular part of the boundary and its normal derivative vanishes on the straight parts of the boundary. Find the potential by the method of images and by an expansion in $\varphi$.

6.41 Let $R$ denote the curvilinear quadrilateral
\[ a < r < b; \quad 0 < \varphi < \alpha. \]

Find the Green's function for the negative Laplacian in $R$, first by an expansion in $\varphi$, then by an expansion in $r$. 
6.42 Find the Green's function for the interior of the unit sphere in three dimensions by an expansion in spherical harmonics. First, place the source at \( r = r_0, \theta = 0 \); by using (A.12) show that

\[
g = \frac{1}{4\pi|x - \xi|} - \sum_{n=0}^{\infty} \frac{(rr_0)^n}{4\pi} P_n(\cos \theta).
\]

Show that this is equivalent to (6.102). Find \( g \) for an arbitrary position of the source.

6.43 Consider the two-dimensional wedge of face angle \( \alpha \), that is, the region \( R \) described in polar coordinates by \( 0 < r < \infty, 0 < \phi < \alpha \). A two-dimensional point source (that is, one with a logarithmic potential) is placed at \( r_0, \phi_0 \). The sides of the wedge are kept at zero potential.

(a) If \( \alpha = \pi/n \), where \( n \) is a positive integer, the solution can be found by the method of images. Carry out the calculation for the case \( n = 2 \). The method of images can be extended to arbitrary \( \alpha \) by an ingenious method due to Sommerfeld (Section 7.12).

(b) For arbitrary \( \alpha \), find the solution by expanding in the \( \phi \) eigenfunctions (which are \( \{\sin(n\pi\phi/\alpha)\} \)). Obtain the result

\[
g = \sum_{n=1}^{\infty} \frac{1}{n\pi} \sin \frac{n\pi\phi_0}{\alpha} \sin \frac{n\pi\phi}{\alpha} \frac{r_{n\pi/\alpha}}{r_{<} r_{>}}. \tag{6.133}
\]

Specialize to the case of a half-plane (\( \alpha = \pi \)) and check (6.106).

(c) The solution can also be obtained by a radial expansion. The radial equation found by separating variables for Laplace's equation is

\[-(r R')' - \frac{\lambda R}{r} = 0, \quad 0 < r < \infty.\]

This eigenvalue problem leads to the Mellin transform (4.107) and its inverse (4.108).

Define the Mellin transform of the Green's function as

\[G_M(v) = \int_0^\infty gr^{-iv-1} \, dr,\]

then

\[g = \frac{1}{2\pi} \int_{-\infty}^{\infty} r^{iv} G_M(v) dv,\]

where the dependence on \( \phi \) has been suppressed.

The differential equation for \( g \) is

\[-\frac{\partial}{\partial r} \left( r \frac{\partial g}{\partial r} \right) - \frac{1}{r} \frac{\partial^2 g}{\partial \phi^2} = \delta(r - r_0) \delta(\phi - \phi_0).\]
Multiply by $r^{-iv}$ and integrate from $r = 0$ to $r = \infty$. This yields an ordinary differential equation for $G_M$. Solve this equation to find

$$G_M = \frac{r_0^{-iv}}{v \sinh v\alpha} \sinh v\varphi_0 \sinh v(\alpha - \varphi_0),$$

$$g = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{(r/r_0)^{iv}}{v \sinh v\alpha} \sinh v\varphi_0 \sinh v(\alpha - \varphi_0).$$

By contour integration show that this result can be transformed into (6.133).

6.44 (a) Show that the function

$$w = e^{ix} \frac{z - z_0}{z - \bar z_0}$$

is the mapping function for the upper half-plane. Hence show that the Green's function for the upper half-plane is

$$-\frac{1}{2\pi} \log \frac{|z - z_0|}{|z - \bar z_0|},$$

which confirms (6.106).

(b) If $s(z)$ maps the region $R$ into the upper half-plane, show that the Green's function for $R$ is

$$-\frac{1}{2\pi} \log \frac{|s(z) - s(z_0)|}{|s(z) - \bar s(z_0)|}. \quad (6.134)$$

6.45 Consider the two-dimensional problem of the Green's function for a semiinfinite strip

$$0 < y < \infty, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

(a) Find $g$ by the method of images.

(b) Find $g$ by an expansion in the $x$ eigenfunctions.

(c) Find $g$ by a sine transform on $y$. The relevant transform pair was derived in Chapter 4; see (4.60) and (4.61). For our purposes we define the sine transform $F_s(v)$ of $f(y)$ by

$$F_s(v) = \int_0^\infty f(y) \sin vy \, dy,$$

with the inversion

$$f(y) = \frac{2}{\pi} \int_0^\infty F_s(v) \sin vy \, dv.$$

(d) Find $g$ by using (6.134). [Consider $s(z) = \sin z$.]
6.46 (a) By using the divergence theorem, show that the Neumann problem

$$-\nabla^2 u = q(x), \quad x \text{ in } R; \quad \frac{\partial u}{\partial n} = f, \quad x \text{ on } \sigma,$$

can have a solution only if $f$ and $q$ are related by

$$\int_R q(x)dx = -\int_\sigma f(x)dS_x. \quad (6.135)$$

This consistency condition has a simple physical meaning in steady-state heat conduction; the steady state can be achieved only if the net heat input from body sources is equal to the net heat flow through the boundary.

(b) The problem

$$-\nabla^2 g = \delta(x - \xi), \quad x, \xi \text{ in } R; \quad \frac{\partial g}{\partial n} = 0, \quad x \text{ on } \sigma$$

cannot have a solution, since the consistency condition (6.135) is not satisfied. Instead we can construct a modified influence function as in Section 1.6. Show that the consistency condition is satisfied for the problem

$$-\nabla^2 h = \delta(x - \xi) - \frac{1}{V}, \quad x, \xi \text{ in } R; \quad \frac{\partial h}{\partial n} = 0, \quad x \text{ on } \sigma, \quad (6.136)$$

where $V$ is the volume of $R$.

(c) Construct the solution of (6.136) by a Fourier expansion on $\varphi$, when $R$ is the unit circle. Find the solution

$$h(r, \varphi | r_0, \varphi_0) = A + \frac{r^2}{4\pi} - \frac{1}{2\pi} \log r_\ast + \sum_{n=1}^{\infty} \cos \frac{n(\varphi - \varphi_0)}{2\pi n} r^n(r^n_\ast + r^n_\ast), \quad (6.137)$$

where $A$ is an arbitrary constant and

$$r_\ast = \max (r, r_0), \quad r_\ast = \min (r, r_0).$$

6.8 SOME PHYSICAL APPLICATIONS OF POTENTIAL THEORY

Charged Conductor and Electrostatic Capacity

Let $R_i$ be a bounded region in three dimensions with boundary $\sigma$ and let $R_e$ be its exterior. Viewing $\sigma$ as a metallic shell, we shall be interested in finding the potential due to the presence of a charge $Q'$ placed on the conductor. The charge distributes itself on $\sigma$ so that $\sigma$ is an equipotential surface; since the charge is distributed in a finite portion of space, we expect the corresponding
potential $u'$ to vanish at infinity. Although the constant value of the potential on the surface is unknown, we will solve the problem when the potential on $\sigma$ is $u = 1$. We can then calculate the corresponding charge $Q$ on $\sigma$ from $Q = -\int_{\sigma} (\partial u/\partial n)dS$. By multiplying $u$ by the ratio $Q'/Q$, we will have the solution $u'$ of the original problem. In the calculation of the capacity $C$ this last step can be omitted entirely, for the capacity is defined as the ratio of the charge on the conductor to the potential of the conductor, and clearly

$$C = \frac{Q'}{(u')_{\sigma}} = \frac{Q}{(u)_{\sigma}} = Q.$$  \hfill (6.138)

We therefore consider the exterior Dirichlet problem

$$\nabla^2 u = 0, \quad x \text{ in } R^3; \quad u = 1 \text{ on } \sigma, \quad u_{\mid \sigma} = 0.$$ \hfill (6.139)

One should observe that the interior problem for these boundary values leads to the solution $u_i = 1$; the exterior problem is much harder! For special geometries such as a sphere, the solution can be obtained by separation of variables, but in general all one can do is to reduce the problem to an integral equation for the charge density on $\sigma$. According to (6.66), we have

$$1 = \int_{\sigma} \frac{1}{4\pi|s-\xi|} I(\xi)dS_\xi, \quad \text{for all } s \text{ on } \sigma.$$ \hfill (6.140)

Here

$$I(\xi) = \frac{\partial u_i}{\partial n} - \frac{\partial u}{\partial n},$$

but since $u_i = 1$, $\partial u_i/\partial n = 0$, and

$$I(\xi) = -\frac{\partial u}{\partial n},$$

where $-\partial u/\partial n$ is the unknown charge density on $\sigma$.

Another integral equation (of the second kind) is available to determine $I$. We differentiate the expression for $u(x)$ with respect to the normal direction $n$ at some fixed point $s$ at the surface. Then

$$\frac{\partial u}{\partial n}(x) = \int_{\sigma} \frac{\partial}{\partial n} E(x \mid \xi) I(\xi)dS_\xi.$$  

As $x \to s$, the left side approaches $-I(s)$ and the right side can be calculated from (6.43). We then have

$$-I(s) = -\frac{1}{2}I(s) + \int_{\sigma} \frac{\cos(\xi - s, v)}{4\pi|s - \xi|^2} I(\xi)dS_\xi$$

or

$$-\frac{I(s)}{2} = \int_{\sigma} \frac{\cos(\xi - s, v)}{4\pi|s - \xi|^2} I(\xi)dS_\xi.$$ \hfill (6.140a)
It is clear that the integral operator generated by the kernel in the above integral equation must have $-\frac{1}{2}$ for an eigenvalue. The eigenfunction $I(\xi)$ is then determined only to a multiplicative constant which must be adjusted so that the total charge on the conductor is 1.

A special case of some interest is the problem of a conductor shaped like an arbitrary open surface (see Figure 6.5). In this case the derivation which led to (6.140) needs a very simple modification since there is no associated interior problem. Let us call one side of $\sigma$ the positive side and denote it by $\sigma_+$ (the other side is labeled $\sigma_-$). The normal, pointing away from the surface, on $\sigma_+$ is denoted by $n$.

![Figure 6.5](image)

Our Dirichlet problem is, then,

$$\nabla^2 u = 0, \quad x \text{ not on } \sigma; \quad u = 1 \text{ on } \sigma, \quad u|_\infty = 0.$$ 

The free-space fundamental solution satisfies

$$-\nabla^2 E = \delta(x - \xi), \quad \text{for all } x, \xi; \quad E|_\infty = 0.$$ 

Explicitly, $E = 1/4\pi|x - \xi|$.

Multiply the differential equation for $u$ by $E$, the one for $E$ by $u$, add and integrate over a region bounded externally by a large sphere $\sigma_r$ and internally by $\sigma_+$ and $\sigma_-$. Using Green's theorem, we have

$$u(\xi) = \int_{\sigma_r} \left( E \frac{\partial u}{\partial n} - u \frac{\partial E}{\partial n} \right) ds_x + \int_{\sigma_+} \left( u \frac{\partial E}{\partial n} - E \frac{\partial u}{\partial n} \right) ds_x$$

$$+ \int_{\sigma_-} \left( E \frac{\partial u}{\partial n} - u \frac{\partial E}{\partial n} \right) ds_x.$$ 

The contribution from $\sigma_r$ vanishes as $r \to \infty$ because of the behavior of $u$ and $E$ ([see (6.60)]. The functions $E$ and $\partial E/\partial n$ have their only singularity at the point $\xi$ in space, and $\xi$ is required not to be on $\sigma$. Thus $E$ and $\partial E/\partial n$ have the same values on either side of $\sigma$. The function $u$ takes on the same value 1 on $\sigma_+$ and $\sigma_-$, but $\partial u/\partial n$ may have different values on $\sigma_+$ and $\sigma_-$. Therefore,

$$u(\xi) = \int_{\sigma} E(x|\xi) \left[ -\frac{\partial u}{\partial n}(x+) + \frac{\partial u}{\partial n}(x-) \right] ds_x.$$
Since \( \frac{\partial u}{\partial n}(x+) \) is the charge density on \( \sigma_+ \) and \( \frac{\partial u}{\partial n}(x-) \) is the charge density on \( \sigma_- \), the bracketed quantity is the total charge density \( I(x) \) on \( \sigma \). Thus we can write

\[
u(\xi) = \int_{\sigma} E(x | \xi) I(x)dS_x;
\]

renaming the variables and using the symmetry of \( E \), we find

\[
u(x) = \int_{\sigma} E(x | \xi) I(\xi)dS_\xi.
\]

Now let \( x \) approach a point \( s \) on \( \sigma \) from either side of \( \sigma \); then, by the continuity of simple layer potentials, we have

\[
1 = \int_{\sigma} E(s | \xi) I(\xi)dS_\xi, \quad \text{for all } s \text{ on } \sigma. \tag{6.141}
\]

This integral equation determines \( I(\xi) \), the total charge density on \( \sigma \). On the other hand, the approach which led to (6.140a) now fails entirely; indeed we have

\[
\frac{\partial u}{\partial v}(x) = \int_{\sigma} \frac{\partial E(x | \xi)}{\partial v} I(\xi)dS_\xi,
\]

and, letting \( x \) approach \( s^\pm \), we obtain the two equations

\[
\frac{\partial u}{\partial v}(s+) = -\frac{1}{2}I(s) + \int_{\sigma} \frac{\partial E(s | \xi)}{\partial v} I(\xi)dS_\xi;
\]

\[
\frac{\partial u}{\partial v}(s-)= \frac{1}{2}I(s) + \int_{\sigma} \frac{\partial E(s | \xi)}{\partial v} I(\xi)dS_\xi.
\]

By subtracting the two equations we find that

\[
I(s) = \frac{\partial u}{\partial v}(s-) - \frac{\partial u}{\partial v}(s+),
\]

which is merely the definition of \( I \). By adding the two equations we get an integral equation in two unknowns, which of course is of no help. Thus, in this case, we must fall back on (6.141).

**Charged Conductor in Two Dimensions**

Suppose a cylindrical conductor carries a charge \( Q \) per unit length; without loss of generality we take \( Q = 1 \). As usual, the surface of the conductor is an equipotential surface, but of course we do not know the value of this constant potential. We shall take the cross section of the cylinder to be a closed curve \( C \) whose exterior will be denoted by \( R_c \). Our problem is clearly two-dimensional and we may therefore carry out the analysis in a fixed plane perpendicular to
the cylinder. We are looking for a function harmonic in \( R_e \), constant on \( C \), and such that

\[
- \int_C \frac{\partial u}{\partial n} \, dl = 1.
\]

Since \( \nabla^2 u = 0 \) in \( R_e \), we have also

\[
- \int_{C_r} \frac{\partial u}{\partial n} \, dl = 1,
\]

where \( C_r \) is any circle enclosing \( C \). On and outside \( C_r \), \( u \) has the representation

\[
\sum_{n=-\infty}^{\infty} a_n r^n e^{i n \varphi} + \sum_{n=-\infty}^{\infty} b_n r^n e^{-i n \varphi} + A \log r.
\]

In the calculation of the integral (6.142), the sums do not contribute and therefore we must have \( A = -1/2\pi \). Moreover, we expect the potential at infinity to behave somewhat as if it were generated by a two-dimensional unit source at the origin. We therefore postulate the absence of all positive powers of \( r \) in (6.143). It is now clear that we can choose arbitrarily the value of the constant potential on \( C \) and we shall therefore choose \( u | C = 0 \); this choice does not imply that \( a_0 \) vanishes in (6.143). We are now in a position to formulate the problem of the charged conductor in two dimensions:

\[
\nabla^2 u = 0, \quad x \text{ in } R_e; \quad u | C = 0, \quad \lim_{r \to \infty} \frac{u}{\log r} = -\frac{1}{2\pi}.
\]

The boundary value problem (6.144) has one and only one solution. We now derive an integral equation for the charge density on \( C \). We shall need the free-space fundamental solution \( E = -(1/2\pi) \log |x - \xi| \), which satisfies

\[
-\nabla^2 E = \delta(x - \xi), \quad \text{for all } x \text{ and } \xi.
\]

With \( \xi \) a fixed point in \( R_e \), we obtain by what is now a familiar procedure

\[
u(\xi) = \int_{C_r} \left( E \frac{\partial u}{\partial n} - u \frac{\partial E}{\partial n} \right) dl_x + \int_C \left( u \frac{\partial E}{\partial n} - E \frac{\partial u}{\partial n} \right) dl_x,
\]

where \( C_r \) is a circle of radius \( r \) enclosing \( C \). As \( r \to \infty \), we can calculate the first integral, but, unlike the three-dimensional case, it does not vanish. With \( \xi \) fixed and \( |x| \) large, we have, using the center of \( C_r \) as the origin,

\[
E = -\frac{1}{2\pi} \log |x - \xi| = -\frac{1}{2\pi} \log r + \frac{1}{2\pi} \frac{|\xi|}{r} \cos \varphi + \cdots,
\]

\[
\frac{\partial E}{\partial n} = \frac{\partial E}{\partial r} = -\frac{1}{2\pi r} - \frac{1}{2\pi} \frac{|\xi|}{r^2} \cos \varphi - \cdots,
\]

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where \( \phi \) is the angle between the vectors \( x \) and \( \xi \) referred to the origin. For \( u \) and \( \partial u/\partial n \) we have
\[
\begin{align*}
  u &= -\frac{1}{2\pi} \log r + c_0 + \frac{c_1}{r} e^{i\phi} + \frac{c_{-1}}{r} e^{-i\phi} + \cdots, \\
  \frac{\partial u}{\partial n} &= \frac{\partial u}{\partial r} = -\frac{1}{2\pi r} - \frac{c_1}{r^2} e^{i\phi} - \frac{c_{-1}}{r^2} e^{-i\phi} - \cdots.
\end{align*}
\]
Since \( dl_x = r d\phi \), we can easily calculate the limiting value of the integral over \( C_r \) and we find
\[
\lim_{r \to \infty} \int_{C_r} \left( E \frac{\partial u}{\partial n} - u \frac{\partial E}{\partial n} \right) dl_x = c_0.
\]
We observe, in passing, that the same result would have been obtained if we had taken \( u \) to behave as \( Q \log r \) at infinity. Consequently, from (6.145),
\[
  u(\xi) = c_0 - \int_C E(x \mid \xi) \frac{\partial u}{\partial n} (x) dl_x, \quad \xi \text{ in } R_e
\]
or
\[
  u(x) = c_0 + \int_C E(x \mid \xi) I(\xi) dl_\xi, \quad x \text{ in } R_e, \quad (6.146)
\]
where \( I(\xi) = \partial u(\xi)/\partial n \) is the charge density on \( C \).

Letting \( x \) approach the boundary \( C \) and using the continuity property of simple layer potentials,
\[
  c_0 = \int_C \frac{1}{2\pi} \log |s - \xi| I(\xi) dl_\xi, \quad s \text{ on } C. \quad (6.147)
\]
The integral equation has two unknowns, the charge density \( I \) and the constant \( c_0 \); but we have not used all the information at our disposal. We must still require that the total charge on the conductor be 1. Two possibilities arise. If the homogeneous equation corresponding to (6.147) has only the trivial solution, then (6.147) will have a unique solution \( I \) proportional to \( c_0 \). We then choose \( c_0 \) such that
\[
  \int_C I(x) dl_x = 1. \quad (6.148)
\]
If instead the homogenous equation has a nontrivial solution, (6.147) will have a solution only if \( c_0 = 0 \), and then \( I \) is determined only to a constant multiple. Therefore we can again use (6.148) to find this constant multiple and hence the charge density on \( C \).

Just as for the three-dimensional case we can also obtain an integral equation of the second kind for \( I \). Differentiate (6.146) with respect to \( \nu \) (the unit
normal at a fixed point \( s \) on \( C \) and let \( x \to s \). Then from (6.51), we find

\[
-I(s) = -\frac{1}{2}I(s) + \int_C k(\xi, s) I(\xi) dl_\xi
\]

or

\[
-\frac{I(s)}{2} = \int_C k(\xi, s) I(\xi) dl_\xi,
\]

where

\[
k(\xi, s) = \frac{\cos (\xi - s, v)}{2\pi|\xi - s|}.
\]

From this point of view \( I \) is an eigenfunction of the integral operator generated by \( k(\xi, s) \), corresponding to the eigenvalue \(-\frac{1}{2}\). The charge density is determined to a multiplicative constant which is then calculated by requiring that the total charge on \( C \) be unity.

If \( C \) is an arc instead of a closed curve, a reasoning similar to the one which led to (6.141) can be used. We obtain

\[
c_0 = \int_C \frac{1}{2\pi} \log |s - \xi| I(\xi) dl_\xi,
\]

where \( I(\xi) \) is the total charge density on \( C \) (that is, the sum of the charge densities on \( C_- \) and \( C_+ \)).

**Examples**

**Example 1.** Charged circular cylinder. Let the radius of the cylinder be \( a \). Then (6.147) becomes

\[
c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \left[ 2a^2 - 2a^2 \cos (\theta - \varphi) \right]^{1/2} I(\varphi) a d\varphi, \quad -\pi < \theta < \pi.
\]

Now

\[
\log \left[ 2a^2 - 2a^2 \cos (\theta - \varphi) \right]^{1/2} = \log a + \frac{1}{2} \log \left[ 2 - 2 \cos (\theta - \varphi) \right]
\]

\[
= \log a - 2 \sum_{n=1}^{\infty} \frac{\cos n\theta \cos n\varphi + \sin n\theta \sin n\varphi}{n}.
\]

If we let

\[
I(\varphi) = \sum_{m=0}^{\infty} I_m \cos m\varphi + \sum_{m=1}^{\infty} I'_m \sin m\varphi,
\]

then (6.150) becomes

\[
\frac{c_0}{a} = I_0 \log a - \sum_{n=1}^{\infty} \frac{I_n \cos n\theta}{n} - \sum_{n=1}^{\infty} \frac{I'_n \sin n\theta}{n}, \quad -\pi < \theta < \pi.
\]
We distinguish between two cases:

**Case 1.** If \( \log a \neq 0 \), that is, \( a \neq 1 \), we can solve for the coefficients of \( I \) regardless of the value of \( c_0 \). In fact,

\[
I_n = I'_n = 0, \; n \neq 0; \quad I_0 = \frac{c_0}{a \log a}.
\]

Therefore the charge density on the conductor is constant and, since the total charge is 1,

\[
\int_{-\pi}^{\pi} aI_0 \, d\theta = 1,
\]

and \( I_0 = 1/2\pi a \), \( c_0 = (\log a)/2\pi \). Substituting in (6.146), we find

\[
u = \frac{1}{2\pi} \log (a/r).
\]

**Case 2.** \( a = 1 \), \( \log a = 0 \). Now (6.151) has a solution only if \( c_0 = 0 \); then \( I_n = I'_n = 0, \; n \neq 0 \), and \( I_0 \) is arbitrary. Requiring that the charge on the conductor be 1, we find \( I_0 = 1/2\pi \) and \( u = -(1/2\pi) \log r \), which agrees with what would be obtained from the results of **Case 1** if we substituted \( a = 1 \).

From the physical standpoint nothing strange happens when \( a = 1 \) and we expect all formulas representing physical quantities to be continuous functions of \( a \). This turns out to be reflected by the mathematical results, although a moment of anxiety arose when we discovered that the mathematical formulation was different for the case \( a = 1 \). One can also use the integral equation of the second kind:

\[
-\frac{I(s)}{2} = \int_C k(\xi, s)I(\xi) \, dl_\xi,
\]

where

\[
k(\xi, s) = \frac{\cos(\xi - s, v)}{2\pi|\xi - s|}.
\]

For the case where \( C \) is a circle, \( k \) is a constant (see Exercise 6.22) and therefore \( I(s) \) is a constant. Since the total charge on \( C \) is 1, we have immediately

\[
I = \frac{1}{2\pi a},
\]

which is, of course, a much simpler way of finding the previous result.

**Example 2.** Strip of width \( 2a \). In this two-dimensional problem; the cross section of the strip in the \( xy \) plane is the segment \(-a < x < a, y = 0\). Equation
(6.149) becomes
\[ c_0 = \frac{1}{2\pi} \int_{-a}^{a} \log |x - \xi| I(\xi) d\xi, \quad -a < x < a. \] (6.152)

Let \( x = at, \xi = a\tau; \) then our integral equation becomes
\[ \frac{2\pi}{a} \left[ c_0 - \frac{1}{2\pi} \log a \int_{-1}^{1} aI(at) d\tau \right] = \int_{-1}^{1} \log |t - \tau| I(at) d\tau, \quad -1 < t < 1. \]

According to (6.192) in Exercise 6.47, this equation has the solution
\[ I(at) = \frac{-1}{\pi \log 2\sqrt{1 - \tau^2}} \left[ \frac{2\pi}{a} \right] \left[ c_0 - \frac{1}{2\pi} (\log a) \int_{-1}^{1} aI(at) d\tau \right]. \]

Since the total charge on the strip is 1, we have
\[ \int_{-1}^{1} aI(at) d\tau = 1, \]
\[ I(at) = \frac{-1}{\pi \log 2\sqrt{1 - \tau^2}} \left[ \frac{2\pi}{a} \right] \left[ c_0 - \frac{1}{2\pi} (\log a) \right]. \]

To find \( c_0, \) we integrate both sides with respect to \( \tau \) from \(-1\) to \(1:\)
\[ 1 = \frac{2\pi}{\log 2} \left[ c_0 - \frac{1}{2\pi} \log a \right], \]
or
\[ c_0 = \frac{1}{2\pi} \log \frac{a}{2} \]
and
\[ I(at) = \frac{1}{a\pi\sqrt{1 - \tau^2}} = \frac{1}{\pi\sqrt{a^2 - a^2\tau^2}}; \]
that is,
\[ I(\xi) = \frac{1}{\pi\sqrt{a^2 - \xi^2}}. \] (6.153)

Note that the procedure is correct even when \( c_0 = 0, \) that is, when \( a = 2. \) Equation (6.152) is then a homogeneous equation which has the nontrivial solution (6.153), normalized so that the charge on the strip is unity.

One should note that the charge density becomes infinite at the ends of the strip. This behavior is characteristic of all conductors with edges, as the next example illustrates.
Example 3. Semi-infinite strip. In this two-dimensional problem the cross section of the strip in the $xy$ plane is the semi-infinite segment $0 < x < \infty$, $y = 0$.

The method of integral equations would require modification, since we can no longer dispose so readily of the integral over $C_r$. We cannot expect the behavior of the potential at infinity to be logarithmic because charges are not limited to a finite region of space. In fact, if we have a charged infinite strip: $-\infty < x < \infty$, $y = 0$, the corresponding potential is proportional to $|y|$ and the total charge on the strip is infinite. We obtain the solution for the semi-infinite strip by separation of variables in polar coordinates (for convenience the range of $\phi$ is chosen as $0 < \phi < 2\pi$). The problem is then

$$\nabla^2 u = 0, \quad 0 < r < \infty, \quad 0 < \phi < 2\pi; \quad u(r, 0+) = u(r, 2\pi-) = 0.$$  \hspace{1cm} (6.154)

When solutions of the form $R(r)\Phi(\phi)$ are substituted in (6.154), we find

$$-\Phi'' = \lambda \Phi, \quad \Phi(0) = \Phi(2\pi) = 0,$$  

$$r(rR')' = \lambda R.$$  

The $\phi$ equation has the eigenvalues $\lambda_m = m^2/4, m = 1, 2, \ldots$ and eigenfunctions $\sin (m/2)\phi$. The radial equation then has independent solutions $r^{m/2}$ and $r^{-m/2}$. Thus the separable solutions satisfying the boundary condition on the strip are

$$r^{m/2} \sin \frac{m}{2} \phi, \quad r^{-m/2} \sin \frac{m}{2} \phi; \quad m = 1, 2, \ldots.$$  

On the basis that the potential at infinity is certainly no more singular than for an infinite strip, we reject all powers of $r$ larger than 1.

Let us now look at the charge density on the strip. We have

$$I(r) = -\frac{\partial u}{\partial n}(r, 0+) + \frac{\partial u}{\partial n}(r, 2\pi-),$$

$$\frac{\partial}{\partial n} = \frac{1}{r} \frac{\partial}{\partial \phi}.$$  

If $m$ is even, then $I(r)$ is identically 0, so we must restrict $m$ to odd integral values. The possible charge densities are $-1/r^{1/2}$, $-1/r^{3/2}$, $-1/r^{5/2}$, $\ldots$, corresponding, respectively, to the potentials

$$r^{1/2} \sin \frac{\phi}{2}, \quad r^{-1/2} \sin \frac{\phi}{2}, \quad r^{-3/2} \sin \frac{3\phi}{2}, \quad \ldots.$$  \hspace{1cm} (6.155)

Only the first of these will yield a finite charge on a finite portion of the strip; we therefore exclude all other possible charge densities. We are left with the solution

$$u = Ar^{1/2} \sin \frac{1}{2} \phi$$  \hspace{1cm} (6.156)
as the potential due to a charged semi-infinite strip; we cannot calculate $A$
since the charge on the whole semi-infinite strip is infinite.

Another way of selecting the solution (6.156) among all the possibilities
(6.155) is to require that the electrostatic energy in any finite region should be
finite. The electrostatic energy within a circle of radius $\varepsilon$ about the origin is

$$\int_0^\varepsilon r\,dr \int_0^{2\pi} \,d\phi |\text{grad } u|^2.$$  \hfill (6.157)

Now

$$|\text{grad } u|^2 = \frac{1}{r^2} \left( \frac{\partial u}{\partial \phi} \right)^2 + \left( \frac{\partial u}{\partial r} \right)^2,$$

and (6.155) yields the respective energy densities

$$\frac{1}{r}, \frac{1}{r^3}, \frac{1}{r^5}, \cdots,$$

only the first of which leads to a finite value for (6.157).

**Grounded Conductor in an External Field**

Let $u_0$ be the potential due to a set of charges located in the finite portion of
space or at infinity (the latter case might occur, for instance, if $u_0$ is the potential
$Ax \cdot e$ corresponding to a uniform electrostatic field in the $e$ direction). We
now place a grounded conductor, with boundary $\sigma$, in a portion of space
where no charges are present. The potential of the configuration is no longer
$u_0$, but is altered because of the charges induced on $\sigma$. The region interior to
$\sigma$ will be denoted by $R_i$ and the exterior by $R_e$. We can write the total potential $u$ as the sum

$$u = u_0 + u_s,$$

where $u_s$ is the secondary potential due to the induced charges on $\sigma$. The
function $u_s$ satisfies the exterior boundary value problem

$$\nabla^2 u_s = 0, \quad \text{in } R_e; \quad u_s = -u_0, \quad \text{x on } \sigma; \quad u_s|_{\infty} = 0.$$

By (6.65), we find

$$u_s(x) = \int_{\sigma} E(x|\xi)I(\xi)dS_{\xi},$$  \hfill (6.158)

where

$$I(\xi) = \frac{\partial u_i}{\partial n} - \frac{\partial u_s}{\partial n},$$

and $u_i$ is the solution of the interior Dirichlet problem with boundary value.
Thus \( u_i = -u_0 \), and

\[
I(\xi) = -\frac{\partial u_0}{\partial n} - \frac{\partial u_s}{\partial n} = -\frac{\partial u}{\partial n};
\]

that is, \( I \) is the charge density on the conductor.

As we let \( x \) approach a point \( s \) on the boundary, we obtain, from (6.158), the integral equation

\[
-u_0(s) = \int_\sigma E(s \mid \xi) I(\xi) dS_\xi,
\]

(6.159)

which can be used to determine \( I \).

If we prefer we can find an integral equation of the second kind to determine \( I \). Taking the derivative of (6.158) in the direction \( n \) (the normal to \( \sigma \) at the fixed point \( s \)), we find, on letting \( x \) approach \( s \) and using (6.43),

\[
\frac{\partial u_s}{\partial n}(s) = -\frac{1}{2} I(s) + \int_\sigma I(\xi) \frac{\cos(\xi - s, n)}{4\pi|s - \xi|^2} dS_\xi.
\]

Now \( \frac{\partial u_s}{\partial n}(s) = -\frac{\partial u_0(s)}{\partial n} - I(s) \), so that

\[
-\frac{\partial u_0}{\partial n}(s) = \frac{I(s)}{2} + \int_\sigma I(\xi) \frac{\cos(\xi - s, n)}{4\pi|s - \xi|^2} dS_\xi.
\]

(6.160)

The left side is known (in fact, it just the normal component of the external electric field on \( \sigma \)). We have therefore an inhomogeneous integral equation of the second kind for \( I \). Its disadvantage is that it must be solved in a case where the corresponding homogeneous equation has a nontrivial solution [see (6.140a)].

**EXAMPLE**

A grounded metallic sphere is introduced in an external field whose potential is \( u_0 \). We want to find the charge density induced on the conductor. For a sphere of radius \( a \), we have

\[
\cos(\xi - s, n) = -\frac{|s - \xi|}{2a},
\]

so that the integral equation (6.160) becomes

\[
-\frac{\partial u_0}{\partial n}(s) = \frac{I(s)}{2} - \int_\sigma \frac{1}{8\pi a|s - \xi|} I(\xi) dS_\xi,
\]

(6.161)

whereas the integral equation (6.159) is

\[
-u_0(s) = \int_\sigma \frac{1}{4\pi|s - \xi|} I(\xi) dS_\xi.
\]

(6.162)
§6.8] SOME PHYSICAL APPLICATIONS OF POTENTIAL THEORY

Each integral equation could now be solved by using an expansion in spherical harmonics as in (6.67), but a much simpler procedure happens to be available in our case. Multiply (6.161) by $2a$ and add to (6.162) to obtain

$$aI(s) = -2a \frac{\partial u_0}{\partial n}(s) - u_0(s),$$

or

$$I(s) = -2 \frac{\partial u_0}{\partial n}(s) - \frac{u_0(s)}{a}.$$

We can specialize this to the case of a unit point source located at a distance $r_0(>a)$ from the center of the sphere. Then, if $R$ is the distance from the point $s$ on the surface to the point source, we have

$$u_0(s) = \frac{1}{4\pi R},$$

$$\frac{\partial u_0}{\partial n}(s) = -\frac{1}{4\pi R^3} \left[ \frac{R^2 - r_0^2}{2} + \frac{a^2}{2} \right],$$

the latter result requiring a straightforward calculation. Thus

$$I(s) = -\frac{1}{4\pi R^3} [r_0^2 - a^2],$$

which could also have been obtained by the method of images.

**Steady Heat Conduction in a Composite Medium**

Let $R$ be a region in space consisting of two contiguous parts $R_1$ and $R_2$. The part $R_1$ is filled with a homogeneous material of thermal conductivity $k_1$, whereas $R_2$ is occupied by another homogeneous material of thermal conductivity $k_2$. The boundary of $R$ is denoted by $\sigma$ and the interface between $R_1$ and $R_2$ by $B$. We want to investigate typical boundary value problems in steady heat conduction for the composite medium $R$. If no sources are present, the temperature $u$ must satisfy $\nabla^2 u = 0$ in each of the parts $R_1$ and $R_2$; in addition to the usual boundary conditions on $\sigma$ (say, the temperature is given on $\sigma$), we shall have to impose certain matching conditions on the interface $B$. To formulate these matching conditions we must appeal to the physics of the problem. We have postulated the absence of sources in $R_1$ and $R_2$; if we further assume that no surface sources are created on the interface $B$, the net heat flow through any closed surface $S$ (whose interior lies entirely in $R$) must vanish. If we choose for $S$ a small pillbox enclosing a point $s$ on $B$, we find

$$k_2 \frac{\partial u_2}{\partial n}(s) = k_1 \frac{\partial u_1}{\partial n}(s), \quad s \text{ on } B,$$

(6.163)
where \( n \) is the normal to \( B \) pointing from \( R_1 \) to \( R_2 \), and \( u_1 \) and \( u_2 \) refer to the temperatures in \( R_1 \) and \( R_2 \), respectively.

A second matching condition is obtained by requiring that the temperature be continuous at the interface; this condition is reasonable on physical grounds when \( R_1 \) and \( R_2 \) are in intimate contact at the interface. Thus we shall require

\[
u_2(s) = u_1(s), \quad s \text{ on } B. \tag{6.164}
\]

The matching conditions (6.163) and (6.164) can also be deduced from another point of view, which has considerable mathematical merit. Instead of a composite medium, consider an inhomogeneous medium with a continuously varying thermal conductivity \( k(x, \varepsilon) \), where \( \varepsilon \) is a positive parameter and

\[
\lim_{\varepsilon \to 0^+} k(x, \varepsilon) = \begin{cases} k_1, & x \text{ in } R_1; \\ k_2, & x \text{ in } R_2. \end{cases} \tag{6.165}
\]

Since any piecewise continuous function can be approximated by infinitely differentiable functions, an appropriate sequence \( k(x, \varepsilon) \) with the property (6.165) can surely be found. We can then solve the boundary value problem for the composite medium as follows: First, solve the boundary value problem for the inhomogeneous medium with conductivity \( k(x, \varepsilon) \), \( \varepsilon > 0 \). Here we impose the boundary condition on \( \sigma \) but no conditions on \( B \) [of course, the solution \( u(x, \varepsilon) \) will be continuous and have continuous derivatives on \( B \), since \( B \) does not play any special role in this problem]. Moreover, \( u \) satisfies the differential equation

\[
\nabla \cdot (k \nabla u) = 0, \quad x \text{ in } R. \tag{6.166}
\]

After finding the unique solution \( u(x, \varepsilon) \), we then take the limit as \( \varepsilon \to 0^+ \) to obtain the solution of the problem for the composite medium. It can be proved that this procedure leads to exactly the same solution as the one resulting from applying the matching conditions (6.163) and (6.164). As an illustration, consider the problem of heat conduction in a rod, \(-1 < x < 1\), with boundary temperatures \( u(-1) = 0 \), \( u(1) = 1 \). The thermal conductivity is a positive, continuous function \( k(x, \varepsilon) \). Our boundary value problem is

\[
\frac{d}{dx} \left( k \frac{du}{dx} \right) = 0, \quad -1 < x < 1, \quad u \big|_{x=-1} = 0, \quad u \big|_{x=1} = 1.
\]

We find explicitly

\[
u(x, \varepsilon) = \left[ \int_{-1}^{1} k^{-1}(x, \varepsilon) \, dx \right]^{-1} \int_{-1}^{x} \frac{1}{k(x, \varepsilon)} \, dx.
\]

Now suppose that \( k(x, \varepsilon) \) has the property (6.165), where \( R_1 \) is \(-1 < x < 0\) and \( R_2 \) is \( 0 < x < 1 \). This can be accomplished, for instance, by choosing

\[
k(x, \varepsilon) = \frac{k_1 + k_2}{2} + \frac{k_2 - k_1}{\pi} \arctan \frac{x}{\varepsilon}.
\]
As \( \varepsilon \to 0^+ \), we find
\[
\begin{align*}
u(x) &= \left[ \frac{k_1 k_2}{k_1 + k_2} \right] \left( \frac{(x + 1)/k_1}{(1/k_1) + (x/k_2)} \right), & -1 < x < 0; \\
&= \left( \frac{k_1 k_2}{k_1 + k_2} \right) \left( \frac{(1/k_1) + (x/k_2)}{k_1 k_2} \right), & 0 < x < 1.
\end{align*}
\]

(6.167)

It is now clear from (6.167) that \( u(x) \) satisfies
\[
\frac{d^2 u}{dx^2} = 0; \quad -1 < x < 0, \quad 0 < x < 1, \quad u(-1) = 0, \quad u(1) = 1 \quad (6.168)
\]

and the matching conditions
\[
u(0-) = u(0+),
\]
\[
k_1 u'(0-) = k_2 u'(0+).
\]

This is, of course, just the direct formulation of the problem of a composite medium with the matching conditions (6.163) and (6.164). This direct formulation is much more manageable since we only have to solve Laplace's equation instead of the more complicated equation (6.166).

In fact, starting from (6.168), we find
\[
\begin{align*}
u(x) &= \begin{cases} A(x + 1), & -1 < x < 0, \\
1 + B(x - 1), & 0 < x < 1.
\end{cases}
\end{align*}
\]

Applying the matching conditions, we have \( A = k_2/k_1 + k_2, \quad B = k_1/k_1 + k_2 \). Our solution then reduces to (6.167).

**Flow of an Ideal Fluid Past an Obstacle**

A rigid obstacle is placed in a plane-parallel flow of an ideal fluid. The flow will, of course, be disturbed by the presence of the obstacle since the normal component of fluid velocity at the surface of the obstacle must vanish. The undisturbed plane parallel flow will be taken as having unit velocity in the \( x \) direction (where \( x \) is a Cartesian coordinate). The corresponding velocity potential of the undisturbed flow is
\[
u_t = x.
\]

The potential \( u \) of the disturbed flow is the sum of \( u_t \) and of the secondary potential \( u_s \) created by the presence of the obstacle. The interior of the obstacle (where no fluid flows) is denoted by \( R_1 \); the boundary of the obstacle is \( \sigma \) and the flow region is the exterior region \( R_e \) complementary to \( R_1 \). We can formulate our problem in terms of the total potential \( u \) or the secondary potential \( u_s \). In the first formulation, we have
\[
\nabla^2 u = 0, \quad x \in R_e; \quad \frac{\partial u}{\partial n} = 0, \quad x \text{ on } \sigma; \quad u \big|_{\infty} = u_t, \quad (6.169)
\]
whereas, in terms of \( u_s \), we can write

\[
\nabla^2 u_s = 0, \quad x \text{ in } R_e; \quad \frac{\partial u_s}{\partial n} = -\frac{\partial u_i}{\partial n}, \quad x \text{ on } \sigma; \quad u_s|_\infty = 0. \quad (6.170)
\]

The latter formulation is a standard exterior Neumann problem for \( u_s \), since we have known values of the normal derivative of \( u_s \) on \( \sigma \) and \( u_s \) vanishes at infinity. On the other hand, (6.169) involves an awkward boundary condition at infinity and is not a standard exterior Neumann problem.

Turning to (6.170) we pause to observe that the condition \( u_s|_\infty = 0 \) can be somewhat strengthened. At large distances from the obstacle, \( u_s \) has the canonical representation of all harmonic functions vanishing at infinity, that is,

\[
u_s = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{mn} r^{-n-1} Y_n^m(\theta, \varphi).
\]

In the secondary flow the amount of fluid crossing a closed surface \( \sigma_r \) at infinity is

\[
\int_{\sigma_r} \frac{\partial u_s}{\partial n} dS = \int_{\sigma} \frac{\partial u_i}{\partial n} dS = -\int_{\sigma} \frac{\partial u_i}{\partial n} dS.
\]

The last integral vanishes since \( u_i \) is harmonic in \( R_i \), the interior region bounded by \( \sigma \). Thus

\[
\int_{\sigma_r} \frac{\partial u_s}{\partial n} dS = 0. \quad (6.171)
\]

Of all the spherical harmonics, only \( Y_0^0 \) has a nonvanishing integral over \( \sigma_r \); therefore, the coefficient \( a_{00} \) must vanish to satisfy (6.171). This means that \( u_s \) actually behaves as \( 1/r^2 \) at infinity, a behavior similar to the potential of a dipole. As expected therefore, the secondary flow may be regarded as due to a system of sinks and sources which neither generates nor extracts a net amount of fluid.

We now attempt to find \( u_s \), the solution of (6.170). By a standard argument using Green's theorem and the fundamental solution \( E = 1/4\pi|x - \xi| \), we find, for \( \xi \) in \( R_e \),

\[
u_s(\xi) = \int_{\sigma_r} \left( E \frac{\partial u_s}{\partial n} - u_s \frac{\partial E}{\partial n} \right) dS_x + \int_{\sigma} \left( u_s \frac{\partial E}{\partial n} - E \frac{\partial u_s}{\partial n} \right) dS_x.
\]

From the behavior of \( u_s \) and \( E \) at infinity we can show that the integral over \( \sigma_r \) goes to 0 as \( r \to \infty \). In the integral over \( \sigma \), we write \( u_s = u - u_i \). The function \( u_i \) is harmonic in the bounded region \( R_i \) and the same is true for \( E \) as long as the source point \( \xi \) is outside \( R_i \). Therefore,

\[
\int_{\sigma} \left( u_i \frac{\partial E}{\partial n} - E \frac{\partial u_i}{\partial n} \right) dS_x = 0, \quad \xi \text{ in } R_e \quad (6.172)
\]

\[
u_s(\xi) = \int_{\sigma} u(x) \frac{\partial E(x|\xi)}{\partial n_x} dS_x, \quad \xi \text{ in } R_e.
\]

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Relabeling the variables and using the symmetry of $E$,

$$u_s(x) = \int_\sigma \frac{\partial E(x|\xi)}{\partial n_\xi} u(\xi) dS_\xi, \quad x \in R_e. \quad (6.173)$$

Thus we have managed to express the solution $u_s(x)$ at any point in the flow field in terms of the unknown values of $u$ on $\sigma$. If we can determine $u$ on $\sigma$ (6.173) will then enable us to find $u_s(x)$ by integration. Letting $x$ approach a point $s$ on the boundary in (6.173), we obtain, from (6.44),

$$u_s(s) = \frac{u(s)}{2} + \int_\sigma \frac{\partial E(s|\xi)}{\partial n_\xi} u(\xi) dS_\xi, \quad s \text{ on } \sigma. \quad (6.174)$$

Since $u_s(s) = u(s) - u_t(s)$, we have the following Fredholm integral equation of the second kind for $u$ on $\sigma$,

$$-u_t(s) = -\frac{u(s)}{2} + \int_\sigma \frac{\partial E(s|\xi)}{\partial n_\xi} u(\xi) dS_\xi, \quad s \text{ on } \sigma. \quad (6.175)$$

One can attempt (unsuccessfully, as it happens) to derive an equation of the first kind for $u$ on $\sigma$ by differentiating (6.173). Let $v$ be the normal to $\sigma$ at $s$; then, from (6.173), we find

$$\frac{\partial u_s(x)}{\partial v} = \frac{\partial}{\partial v} \int_\sigma \frac{\partial E(x|\xi)}{\partial n_\xi} u(\xi) dS_\xi, \quad x \text{ in } R_e. \quad (6.176)$$

Letting $x$ approach $s$, we have

$$-\frac{\partial u_t}{\partial v}(s) = \lim_{x \to s+} \frac{\partial}{\partial v} \int_\sigma \frac{\partial E(x|\xi)}{\partial n_\xi} u(\xi) dS_\xi, \quad s \text{ on } \sigma. \quad (6.176)$$

Unfortunately, the right side cannot be reduced to a pure integral operator acting on $u$. In spite of this, we shall see that (6.176) is of some use in applications dealing with thin open shells. Turning to this case, we denote the two sides of $\sigma$ by $\sigma_+$ and $\sigma_-$, respectively; the normal on $\sigma_+$ pointing away from the surface will be labeled $n$. The appropriate modification to (6.173) is

$$u_s(x) = \int_\sigma \frac{\partial E(x|\xi)}{\partial n_\xi} I(\xi) dS_\xi, \quad (6.177)$$

where

$$I(\xi) = u(\xi +) - u(\xi -) \quad (6.178)$$

is the jump in $u$ across the shell $\sigma$.

If we let $x$ approach $s$, we do not obtain an integral equation for $I$. In fact, we have the two equations

$$u(s+) - u_t(s+) = \frac{I(s)}{2} + \int_\sigma \frac{\partial E(s|\xi)}{\partial n_\xi} I(\xi) dS_\xi,$$

$$u(s-) - u_t(s-) = -\frac{I(s)}{2} + \int_\sigma \frac{\partial E(s|\xi)}{\partial n_\xi} I(\xi) dS_\xi,$$
and by subtraction we only rediscover the definition of $I$, while by addition we obtain the equation

$$u(s+) + u(s-) = 2u_i(s) + 2 \int_\sigma \frac{\partial E(s|\xi)}{\partial n_\xi} I(\xi) dS_\xi. \quad (6.179)$$

Equation (6.179) cannot serve to determine $I$, since the left side is unknown. Still, (6.179) may turn out to be of some use if we want to find the separate values of $u$ on either side of $\sigma$. Supposing that we have managed to solve for $I$ by other means, then (6.178) and (6.179) obviously enable us to calculate $u(s-)$ and $u(s+)$. We are still left with the problem of finding a suitable equation to determine $I(\xi)$. To this end we differentiate (6.177) in the $v$ direction (where $v$ is the normal to the positive side of $\sigma$ at $s$) and obtain

$$\frac{\partial u_s}{\partial v}(x) = \frac{\partial}{\partial v} \int_\sigma \frac{\partial E(x|\xi)}{\partial n_\xi} I(\xi) dS_\xi.$$

As $x \to s\pm$, the left side approaches the given value, $-\frac{\partial u_i(s)}{\partial v}$. Therefore,

$$-\frac{\partial u_i}{\partial v}(s) = \lim_{x \to s\pm} \frac{\partial}{\partial v} \int_\sigma \frac{\partial E(x|\xi)}{\partial n_\xi} I(\xi) dS_\xi, \quad s \text{ on } \sigma. \quad (6.180)$$

Equation (6.180), which is not an integral equation on $\sigma$, can sometimes be used to find $I(\xi)$.

**Remarks.** 1. Equations (6.175) and (6.176) remain valid for the two-dimensional problem of a rigid cylindrical obstacle at right angles to a plane parallel flow. If the obstacle is an open cylindrical shell, (6.180) still applies. Of course, in all cases $E(x|\xi)$ is now the logarithmic kernel $-(1/2\pi) \log |x - \xi|$.

2. If the undisturbed flow $u_i$ is not a plane-parallel flow but is due to some other set of sources at infinity or in the finite portion of space, all our results still apply by simply substituting the undisturbed potential $u_i$ on the left sides of (6.175), (6.176), or (6.180).

**Example**

Normal to a parallel flow in the $x$ direction we place a rigid cylindrical shell whose cross section is the circular arc $C$ defined in polar coordinates by $r = 1$, $-\alpha < \varphi < \alpha$ (see Figure 6.6). We shall reduce the problem of finding the flow field to solving either of two integral equations. Rather than using the Green's function approach, we proceed from more elementary considerations.

Let $u_i$, $u_s$, and $u$ be the undisturbed, secondary, and total potentials, respectively. Then $u_s = u - u_i$, $u_i = x = r \cos \varphi$, and $u_s$ satisfies

$$\nabla^2 u_s = 0, \text{ except when } x \text{ is on } C; \quad (\partial u_s/\partial n)_C = -\cos \varphi; \quad (6.181)$$

$$u_s|_{\infty} = 0.$$
Since both \( u_i \) and the obstacle are symmetric about the \( x \) axis, \( u_s \) and \( u \) will be even functions of \( y \). Thus we can write

\[
u_s(r, \varphi) = \begin{cases} 
\sum_{n=1}^{\infty} b_n r^{-n} \cos n\varphi, & r > 1; \\
\sum_{n=0}^{\infty} a_n r^n \cos n\varphi, & r < 1.
\end{cases}
\]

Now \( \partial u_s(r, \varphi)/\partial r \) is clearly continuous except possibly on \( C \), but there \( \partial u_s(1+, \varphi)/\partial r \) and \( \partial u_s(1-, \varphi)/\partial r \) take on the same value, \(-\cos \varphi\). Hence we have

\[
\frac{\partial u_s}{\partial r} (1+, \varphi) = \frac{\partial u_s}{\partial r} (1-, \varphi), \quad -\pi < \varphi < \pi;
\]

or

\[
\sum_{n=1}^{\infty} (a_n + b_n)n \cos n\varphi = 0, \quad -\pi < \varphi < \pi.
\]

Since the set \{\cos n\varphi\} is a complete orthogonal set for even functions on the interval \(-\pi < \varphi < \pi\), it follows that \( b_n = -a_n \). Thus

\[
u_s(r, \varphi) = -\sum_{n=1}^{\infty} a_n r^{-n} \cos n\varphi, \quad r > 1;
\]

\[
u_s(r, \varphi) = \sum_{n=0}^{\infty} r^n a_n \cos n\varphi, \quad r < 1;
\]

\[
\frac{\partial u_s}{\partial r} (1, \varphi) = \sum_{n=1}^{\infty} n a_n \cos n\varphi;
\]

\[
\frac{\partial u}{\partial r} (1, \varphi) = \cos \varphi + \sum_{n=1}^{\infty} n a_n \cos n\varphi.
\] (6.182)
The first formulation as an integral equation will be in terms of the unknown jump in $u$ on the arc $C$. Let

$$I(\varphi) = u(1+, \varphi) - u(1-, \varphi) = u_a(1+, \varphi) - u_a(1-, \varphi)$$

$$= -a_0 - 2 \sum_{n=1}^{\infty} a_n \cos n\varphi. \quad (6.183)$$

Since $I(\varphi) = 0$, $0 < |\varphi| < \pi$, we have, from the usual formula for Fourier cosine coefficients,

$$a_n = -\frac{1}{\pi} \int_{0}^{\pi} I(\Psi) \cos n\Psi \, d\Psi,$$

$$a_0 = -\frac{1}{\pi} \int_{0}^{\pi} I(\Psi) \, d\Psi.$$  

Moreover, since $\partial u(1, \varphi)/\partial r = 0$, $0 < |\varphi| < \alpha$, we can substitute in (6.182), to obtain

$$\frac{1}{\pi} \sum_{n=1}^{\infty} n \cos n\varphi \int_{0}^{\pi} I(\Psi) \cos n\Psi \, d\Psi = \cos \varphi, \quad 0 < |\varphi| < \alpha. \quad (6.184)$$

Both sides being even functions of $\varphi$, it suffices to let $\varphi$ take on values between 0 and $\alpha$. The presence of the term $n$ in the summation on the left side of (6.184) suggests that the series might diverge; in fact, $\int_{0}^{\pi} I(\Psi) \cos n\Psi \, d\Psi$ is the Fourier coefficient of an even function which vanishes for $\alpha < \varphi < \pi$ and it is only if $I(\alpha-) = 0$ that the coefficients decrease sufficiently fast so as to make (6.184) convergent. Fortunately, the function $I$ for which we are looking has the property $I(\alpha-) = 0$, so that the left side of (6.184) will make sense.

At this stage it is tempting to interchange summation and integration on the left side of (6.184); we then find

$$\int_{0}^{\pi} k(\varphi, \Psi) I(\Psi) \, d\Psi = \cos \varphi, \quad 0 < \varphi < \alpha, \quad (6.185)$$

where

$$k(\varphi, \Psi) = \frac{1}{\pi} \sum_{n=1}^{\infty} n \cos n\varphi \cos n\Psi \quad (6.186)$$

is a divergent series. Therefore (6.185) is not a true integral equation. The difficulty can be removed by rewriting (6.184) so that the factor $n$ is in the denominator. This artifice leads to

$$-\frac{1}{\pi} \frac{d^2}{d\varphi^2} \left[ \sum_{n=1}^{\infty} \frac{\cos n\varphi}{n} \int_{0}^{\pi} I(\Psi) \cos n\Psi \, d\Psi \right] = \cos \varphi,$$
or
\[
\frac{d^2}{d\varphi^2} \int_0^\alpha m(\varphi, \Psi)I(\Psi)d\Psi = \cos \varphi, \quad 0 < \varphi < \alpha, \tag{6.187}
\]
where
\[
m(\varphi, \Psi) = -\frac{1}{\pi} \sum_{n=1}^\infty \frac{\cos n\varphi \cos n\Psi}{n} = \frac{1}{2\pi} \left[ \log 2 + \log |\cos \varphi - \cos \Psi| \right],
\tag{6.188}
\]
the last equality being a consequence of (6.192).

The series for \( m \) is now a convergent series, but the equation for \( I \) has become an integrodifferential equation instead of a pure integral equation. Equation (6.187) can also be derived by a careful application of (6.180).

The second formulation as an integral equation will be in terms of the unknown value of \( \partial u(1, \varphi)/\partial r \) on \( \alpha < |\varphi| < \pi \). Setting
\[
J(\varphi) = \frac{\partial u}{\partial r}(1, \varphi),
\]
we note that \( J = 0, \ 0 < |\varphi| < \alpha \). Using (6.182), we have
\[
a_n = \frac{2}{n\pi} \int_\alpha^\pi J(\Psi) \cos n\Psi d\Psi, \quad n \geq 2;
\]
\[
a_1 = -1 + \frac{2}{\pi} \int_\alpha^\pi J(\Psi) \cos \Psi d\Psi.
\]
Moreover, \( u \) is continuous, \( \alpha < |\varphi| < \pi \), so that, from (6.183),
\[
a_0 + 2 \sum_{n=1}^\infty a_n \cos n\varphi = 0, \quad \alpha < |\varphi| < \pi.
\]
Substituting for \( a_n \) from above, we find
\[
\frac{4}{\pi} \sum_{n=1}^\infty \frac{\cos n\varphi}{n} \int_\alpha^\pi J(\Psi) \cos n\Psi d\Psi = 2 \cos \varphi - a_0, \quad \alpha < \varphi < \pi.
\]
Thus we have the integral equation
\[
\int_\alpha^\pi m(\varphi, \Psi)J(\Psi)d\Psi = -\frac{\cos \varphi}{2} + \frac{a_0}{4}, \quad \alpha < \varphi < \pi, \tag{6.189}
\]
where \( m \) is again given by (6.188).

**Exercises**

6.47 Consider the integral equation
\[
\int_{-1}^1 \log |x - y|f(y)dy = g(x), \quad -1 < x < 1, \tag{6.190}
\]
where \( g \) is given and \( f \) is to be found. Set \( x = \cos \alpha, y = \cos \beta, F(\beta) = f(\cos \beta) \sin \beta, G(\alpha) = g(\cos \alpha) \), to reduce (6.190) to

\[
\int_0^\pi \log |\cos \alpha - \cos \beta| F(\beta) d\beta = G(\alpha), \quad 0 < \alpha < \pi. \tag{6.191}
\]

Use (6.22), the half-angle formula, and

\[\cos \alpha - \cos \beta = 2 \sin \frac{1}{2}(\alpha + \beta) \sin \frac{1}{2}(\alpha - \beta),\]

to obtain

\[
\log |\cos \alpha - \cos \beta| = -\log 2 - 2 \sum_{n=1}^\infty \frac{\cos n\alpha \cos n\beta}{n}. \tag{6.192}
\]

Expand \( F \) and \( G \) in cosine series to find the solution of (6.191) as

\[
F(\beta) = -\frac{1}{\pi^2 \log 2} \int_0^\pi G(\alpha) d\alpha - \frac{2}{\pi^2} \sum_{n=1}^\infty n \cos n\beta \int_0^\pi G(\alpha) \cos n\alpha d\alpha. \tag{6.193}
\]

or

\[
F(\beta) = -\frac{1}{\pi^2 \log 2} \int_0^\pi G(\alpha) d\alpha - \frac{1}{\pi^2} \frac{d^2}{d\beta^2} \int_0^\pi G(\alpha) \log |\cos \alpha - \cos \beta| d\alpha. \tag{6.194}
\]

Show from (6.192) that the solution of (6.190) for the case \( g(\alpha) = 1 \) is

\[
f(x) = -\frac{1}{\pi \log 2} \frac{1}{\sqrt{1 - x^2}}. \tag{6.195}
\]

6.48 Find the potential of a charged conducting wedge.

6.49 Consider the potential \( u \) of a charged cylindrical shell whose cross section is the circular arc \( C \) defined in polar coordinates by \( r = 1, \quad -\alpha < \varphi < \alpha. \)

Obtain an integral equation on \( C \) for the total charge density

\[
I(\varphi) = -\frac{\partial u}{\partial r}(1+, \varphi) + \frac{\partial u}{\partial r}(1-, \varphi),
\]

by expanding \( u \) in a cosine series for \( r > 1 \) and \( r < 1 \). Show that this integral equation is equivalent to (6.149).

6.50 Derive an integral equation for the charge density of a conductor placed in an external field, when the total charge \( \varphi \) on the conductor is specified (the conductor will, of course, be at constant potential, but that potential is not known).

6.51 A grounded, conducting, hemispherical shell is placed in a uniform electrostatic field normal to the shell. Derive an integral equation for the charge density by an expansion in Legendre polynomials.
6.52 Let $x$, $y$, and $z$ be Cartesian coordinates. The half-space $z > 0$ is a medium of thermal conductivity $k_1$ and the half-space $z < 0$ is a medium of thermal conductivity $k_2$. A unit point source is located at $(0, 0, \xi)$, where $\xi > 0$. Find the temperature distribution by the method of images. [Hint: Try to represent the temperature for $z > 0$ as being due to the given point source and to an image source of unknown strength at $(0, 0, -\xi)$; also take the temperature for $z < 0$ as being caused by a source of unknown strength at $(0, 0, \xi)$, and apply the matching conditions (6.163) and (6.164). Study and interpret the limiting cases $k_2 = k_1$ and $k_2 = \infty$.]

6.53 The sphere $r < a$ is a medium of thermal conductivity $k_1$, whereas its exterior is of conductivity $k_2$. A point source is placed at $r = r_0$, where $r_0 < a$. By expanding in Legendre polynomials, find the resulting temperature distribution throughout space.
Chapter 7

EQUATIONS OF EVOLUTION

7.1 INTRODUCTION

In this chapter we shall investigate the behavior of time-dependent physical phenomena, where the state of the physical system depends on an $n$-dimensional space position vector $x$ and on a time variable $t$. We can therefore regard the state of the system as a function of $n + 1$ independent variables; alternatively, we may think of $t$ as a parameter, and view the system as passing through a succession of states as time increases. It is this latter point of view which gives rise to the term evolution. Two important illustrations of partial differential equations governing such phenomena are

1. The equation of heat conduction (also known as the diffusion equation)

$$\frac{\partial u}{\partial t} - \nabla^2 u = q(x, t), \quad (7.1)$$

and

2. The wave equation

$$\frac{\partial^2 u}{\partial t^2} - \nabla^2 u = q(x, t). \quad (7.2)$$

Here we have set equal to 1 the thermal diffusivity $a$ in (7.1) and the speed of propagation $c$ in (7.2). As was seen in Chapter 5, simple linear changes in the independent variables will restore these parameters to their original values.

In these examples $q(x, t)$ is a given function, known as the source function or forcing function, and $u(x, t)$ is the response we wish to calculate. The operator $\nabla^2 = \text{div grad}$ acts only on the space coordinates; if Cartesian coordinates $x_1, \ldots, x_n$ are used for the space position vector, then

$$\nabla^2 = \text{div grad} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}.$$
Associated with (7.1) or (7.2) will be certain given additional boundary and initial conditions.

A typical situation for heat conduction is the following. At some initial time, say $t = 0$, the temperature $u$ is given in a bounded homogeneous region $R$ in space. Heat sources of volume density $q(x, t)$ act in $R$ for $t > 0$, and the temperature is given on the boundary $\sigma$ of $R$ for all $t > 0$. We wish to find the temperature $u(x, t)$ in $R$ for all $t > 0$. The region in space-time for which we want to solve (7.1) is therefore a semiinfinite cylinder parallel to the positive $t$ axis, as in Figure 7.1. The temperature $u$ is given on the lateral surface and on the base of the semiinfinite cylinder. The boundary value problem is

$$\frac{\partial u}{\partial t} - \nabla^2 u = q(x, t), \quad \text{in } R, \quad t > 0;$$

Initial condition: $u(x, 0) = f(x)$, \quad where $f$ is given; \quad (7.3)

Boundary condition: $u(x, t) = h(x, t)$, \quad $x$ on $\sigma$, $t > 0$, \quad where $h$ is given.

Other boundary conditions often occur; for instance, instead of giving the schedule of temperature on the boundary we may stipulate the rate of heat flow $\frac{\partial u}{\partial n}$ on $\sigma$. If the body $R$ is immersed in a surrounding medium at
constant temperature \( u_0 \), then heat transfer at the boundary may occur through the mechanisms of radiation, conduction, and convection. The cumulative effect can often be reasonably approximated by the boundary condition

\[
\frac{\partial u}{\partial n} = - \theta(u - u_0),
\] (7.4)

where \( \theta \) is a positive constant describing the heat-transfer characteristics of the interaction.

For the wave equation (7.2), a typical boundary problem involves two initial conditions instead of one. We give both \( u(x, 0) \) and \( \partial u(x, 0)/\partial t \) since the equation is of the second order in time. This problem arises for instance in investigating the small transverse vibrations of a membrane stretched over the plane region \( R \). At time \( t = 0 \) the deflection and the velocity are both stipulated in \( R \); for all \( t > 0 \), the deflection of the boundary \( \sigma \) of \( R \) is given. The membrane is subject to a transverse pressure \( q(x, t) \), in units of force per area, for \( t > 0 \). We wish to find the deflection \( u(x, t) \) for all \( t > 0 \), where \( u \) is the solution of the boundary value problem

\[
\frac{\partial^2 u}{\partial t^2} - \nabla^2 u = q(x, t), \quad x \text{ in } R, \quad t > 0;
\]

Initial conditions: \( u(x, 0) = f_1(x) \), \( \frac{\partial u}{\partial t} (x, 0) = f_2(x) \); (7.5)

Boundary condition: \( u(x, t) = h(x, t) \), \( x \text{ on } \sigma, \quad t > 0 \).

As befits good applied mathematicians, our principal interest will be in the construction of solutions to the problems (7.3) and (7.5), but, to a lesser extent, we shall also be concerned with questions of existence, uniqueness, and continuous dependence on the data (that is, on the source term, the initial conditions, and the boundary condition).

We shall need Green's theorem for the heat operator and for the wave operator [see (5.79) and (5.81)] as applied to the finite cylinder: \( x \in R, \ 0 < t < T \), where \( T \) is a large positive number which, as we shall see in due course, will disappear in the final analysis. The boundary of the cylinder consists (see Figure 7.1) of three parts: the base \( x \in R, \ t = 0 \); the base \( x \in R, \ t = T \); the lateral surface \( x \in \sigma, \ 0 < t < T \). On the base \( t = 0 \), the normal to the cylinder is in the direction \(-e_t \), whereas on the base \( t = T \) it is in the direction \( e_t \). On the lateral surface the normal to the cylinder coincides with the normal \( n \) to the space region \( R \). Thus (5.79) becomes

\[
\int_0^T dt \int_R dx \left[ v \left( \frac{\partial u}{\partial t} - \nabla^2 u \right) - u \left( - \frac{\partial v}{\partial t} - \nabla^2 v \right) \right] = \int_R -(uv)_{t=0} dx + \int_R (uv)_{t=T} dx + \int_0^T dt \int_{\sigma} dS \left( u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right). \quad (7.6)
\]
Similarly, for the wave operator, (5.81) becomes

\[ \int_0^T dt \int_R dx \left[ v \left( \frac{\partial^2 u}{\partial t^2} - \nabla^2 u \right) - u \left( \frac{\partial^2 v}{\partial t^2} - \nabla^2 v \right) \right] \]

\[ = \int_R \left( -v \frac{\partial u}{\partial t} + u \frac{\partial v}{\partial t} \right) dx + \int_R \left( v \frac{\partial u}{\partial t} - u \frac{\partial v}{\partial t} \right) dx \]

\[ + \int_0^T dt \int_{\sigma} dS \left( u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right). \]  \hspace{1cm} (7.7)

The formulas (7.6) and (7.7) are valid for a bounded space region \( R \). If \( R \) is infinite in extent (shaded region in Figure 7.2), we may apply the formulas to the intersection of \( R \) and large sphere \( V_r \) of radius \( r \), and we must substitute \( R \cap V_r \) for \( R \) in (7.6) and (7.7). The integral over \( \sigma \) consists of an integral over \( \sigma' \) and one over \( \sigma'' \). If we let \( r \to \infty \), the contribution from \( \sigma' \) will converge and the one from \( \sigma'' \) will disappear if \( u \) and \( v \) are sufficiently well-behaved at infinity. Thus (7.6) and (7.7) will still hold. In particular, if \( R = R_n \), the entire \( n \)-dimensional space, there is no space boundary integral at all.

FIGURE 7.2

7.2 CAUSAL GREEN’S FUNCTION FOR HEAT CONDUCTION

Let the region \( R \) be at 0 temperature up to time \( t = t_0 \) when a unit of heat is instantaneously liberated at the point \( x = x_0 \). The boundary \( \sigma \) of \( R \) is kept at 0 temperature for all \( t \). The temperature \( g(x, t \mid x_0, t_0) \) in \( R \) is known as the causal Green’s function of the region \( R \) and satisfies the boundary value

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problem
\[\left(\frac{\partial}{\partial t} - \nabla^2\right)g(x, t | x_0, t_0) = \delta(x - x_0)\delta(t - t_0), \quad x, x_0 \text{ in } R, \quad -\infty < t, t_0 < \infty;\]
\[g(x, t | x_0, t_0) = 0, \quad t < t_0, \quad x, x_0 \text{ in } R;\]
\[g(x, t | x_0, t_0) = 0, \quad x \text{ on } \sigma, \quad x_0 \text{ in } R, \quad -\infty < t, t_0 < \infty.\]
\[(7.8)\]

If the region \(R\) is unbounded we further require that \(g\) vanish at \(|x| = \infty\). The analysis will be carried out for a bounded region but also holds for an unbounded region with the proviso just made. We shall postpone all questions of uniqueness and existence for (7.8) and concentrate instead on some of the properties of the solution. The calculation of \(g\) presents no difficulty for \(t < t_0\), since by assumption \(g \equiv 0\) for \(t < t_0\). For \(t > t_0\), \(g\) can equally well be characterized as the solution of the following initial value problem for the homogeneous heat equation:

\[\left(\frac{\partial}{\partial t} - \nabla^2\right)g(x, t | x_0, t_0) = 0, \quad x, x_0 \text{ in } R, \quad 0 < t, t_0 < \infty;\]
\[g(x, t | x_0, t_0) = 0, \quad x \text{ on } \sigma, \quad x_0 \text{ in } R, \quad 0 < t, t_0 < \infty;\]
\[\lim_{t \to t_0^+} g(x, t | x_0, t_0) = \delta(x - x_0), \quad x, x_0 \text{ in } R.\]
\[(7.8a)\]

The equivalence of (7.8) and (7.8a) was shown for the case \(R = R_n\) in Chapter 5. Exactly the same arguments apply for an arbitrary region \(R\).

By making the change of variables \(t' = t - t_0\), in (7.8) we see that
\[g(x, t | x_0, t_0) = g(x, t - t_0 | x_0, 0),\]
\[(7.9)\]
so that we may as well introduce the source in (7.8) at time 0 rather than at an arbitrary time \(t_0\). If the region \(R\) is the whole of \(n\) space, that is, if \(R = R_n\), then \(g\) is the free-space causal fundamental solution \(C\) introduced in Chapter 5. Thus, if \(R = R_n\),
\[g(x, t | x_0, t_0) = C(x, t | x_0, t_0) = \frac{H(t - t_0)}{[4\pi(t - t_0)]^{n/2}} e^{-|x-x_0|^2/4(t-t_0)},\]
\[(7.10)\]
where \(H\) is the Heaviside function. Of course, if \(R\) does not coincide with \(R_n\) it will be much more difficult to calculate \(g\), but we are content now to show how the general problem (7.3) can be solved in terms of \(g\). For this purpose we intend to use (7.6) with \(u\) the solution of (7.3) and \(v\) an appropriate Green's function. We shall obviously need the Green's function for the operator \(-\partial^2/\partial t^2 - \nabla^2\) rather than for \((\partial^2/\partial t^2) - \nabla^2\); moreover, our Green's function will have to vanish for \(t = T\). We are therefore led to consider the adjoint problem
\[ \left( -\frac{\partial}{\partial t} - \nabla^2 \right) g^*(x, t | x_0, t_0) = \delta(x - x_0)\delta(t - t_0), \quad x, x_0 \in R, \quad -\infty < t, t_0 < \infty; \] (7.11)

\[ g^*(x, t | x_0, t_0) \equiv 0, \quad t > t_0, \quad x, x_0 \in R; \]

\[ g^*(x, t | x_0, t_0) = 0, \quad x \text{ on } \sigma, \quad x_0 \in R, \quad -\infty < t, t_0 < \infty. \]

Note that \( g^*(x, t | x_0, t_0) \) satisfies the backward heat equation (that is, with a reversal of sign for the time variable) and that \( g^* \) vanishes identically for \( t \) greater than \( t_0 \). If we insist on a "physical" interpretation, we might say that (7.11) characterizes a problem of heat concentration rather than diffusion. Next we show that

\[ g^*(x, t | x_0, t_0) = g(x_0, t_0 | x, t), \] (7.12)

so that \( g^*(x, t | x_0, t_0) \) can be obtained from the solution of (7.8) by merely interchanging the variables \((x, t)\) and \((x_0, t_0)\). To prove (7.12), use (7.6) with \( u = g(x, t | x_1, t_1), v = g^*(x, t | x_0, t_0), \) and \((0, T)\) replaced by \((T_1, T_2)\), where \( T_1 < t_0, t_1 < T_2 \). It then follows that

\[ g^*(x_1, t_1 | x_0, t_0) = g(x_0, t_0 | x_1, t_1), \]

which is the desired result.

We can now forget all about \( g^* \). The only thing we need to know is that \( g(x_0, t_0 | x, t) \) satisfies (7.11); that is,

\[ \left( -\frac{\partial}{\partial t} - \nabla^2 \right) g(x_0, t_0 | x, t) = \delta(x - x_0)\delta(t - t_0), \quad x, x_0 \in R, \quad -\infty < t, t_0 < \infty; \] (7.13)

\[ g(x_0, t_0 | x, t) \equiv 0, \quad t > t_0, \quad x, x_0 \in R; \]

\[ g(x_0, t_0 | x, t) = 0, \quad x \text{ on } \sigma, \quad x_0 \in R, \quad -\infty < t, t_0 < \infty. \]

Next, apply (7.6) with \( T > t_0 \), \( u \) the solution of (7.3), and \( v = g(x_0, t_0 | x, t) \). Since \( v = 0 \) for \( t > t_0 \), we find

\[
\begin{align*}
  u(x_0, t_0) &= \int_0^{t_0} dt \int_R dx \ g(x_0, t_0 | x, t)q(x, t) + \int_R dx \ g(x_0, t_0 | x, 0) f(x) \\
  &= -\int_0^{t_0} dt \int_\sigma ds_x \frac{\partial g(x_0, t_0 | x_1, t_1)}{\partial n_x} h(x, t).
\end{align*}
\]

The result no longer contains \( T \) and is valid for all \( x_0 \) in \( R \) and all \( t_0 > 0 \). To put our solution in standard form, we interchange the labels \((x, t)\) and \((x_0, t_0)\),

\[
\begin{align*}
  u(x, t) &= \int_0^t dt_0 \int_R dx_0 \ g(x, t | x_0, t_0)q(x_0, t_0) + \int_R dx_0 \ g(x, t | x_0, 0) f(x_0) \\
  &= -\int_0^t dt_0 \int_\sigma ds_0 \frac{\partial g(x, t | x_0, t_0)}{\partial n_0} h(x_0, t_0). \quad (7.14)
\end{align*}
\]
The expression (7.14) clearly shows the nature of the dependence of $u$ on the various parts of the data. In particular, the temperature at time $t$ depends only on the previous history and not on the source or boundary schedule after time $t$. This solution remains valid for unbounded regions, provided that $f$, $g$, and $h$ are sufficiently well behaved at $|x| = \infty$. If $R$ is the whole of $n$ space, $R_n$, then the boundary term involving $h$ is absent, and $g$ is given explicitly by (7.10).

As a special case, consider the initial value problem for an infinite rod when no sources are present; the temperature $u(x, t)$ satisfies

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \quad -\infty < x < \infty, \quad t > 0; \quad u(x, 0) = f(x). \quad (7.15)$$

The causal fundamental solution for $R_1$ is

$$C(x, t \mid x_0, t_0) = \frac{1}{[4\pi(t - t_0)]^{1/2}} e^{-(x-x_0)^2/4(t-t_0)}, \quad t > t_0,$$

so that, by (7.14), we find

$$u(x, t) = \int_{-\infty}^{\infty} \frac{1}{[4\pi t]^{1/2}} e^{-(x-x_0)^2/4t} f(x_0) dx_0. \quad (7.16)$$

It is easy to verify that if $f(x)$ is bounded, $-\infty < x < \infty$, then the integral above converges and is infinitely differentiable for all $t > 0$ and all $x$. By differentiating under the integral sign, we find that $u$ satisfies the homogeneous heat equation for $t > 0$. To show that the initial value $f(x)$ is assumed, we use the fact, proved in Chapter 5, Equation (5.136), that

$$\lim_{t \to 0^+} C(x, t \mid x_0, 0) = \delta(x - x_0),$$

in the distributional sense. Since

$$\int_{-\infty}^{\infty} \frac{1}{(4\pi t)^{1/2}} e^{-(x-x_0)^2/4t} dx_0 = 1, \quad -\infty < x < \infty, \quad t > 0,$$

we have, from (7.16),

$$\max_{-\infty < x < \infty} |u(x, t)| \leq \max_{t > 0} |f(x)|,$$

and the solution $u(x, t)$ depends continuously on the initial data $f(x)$.

We examine two special cases of (7.15). First suppose that $f(x) = \delta(x - \xi)$; then, from (7.16), we have

$$u(x, t) = \frac{1}{(4\pi t)^{1/2}} e^{-(x-\xi)^2/4t},$$

which is just the causal fundamental solution corresponding to a unit source at $(\xi, 0)$. This confirms a result already mentioned in Chapter 5: An initial
temperature $\delta(x - \xi)$ is equivalent to a source term $\delta(x - \xi)\delta(t)$ in the inhomogeneous equation.

Next suppose that $f(x) = -\delta'(x)$, which corresponds to a unit dipole at $t = 0$. Then, from (7.16), we find

$$
u(x, t) = -\int_{-\infty}^{\infty} \frac{1}{[4\pi t]^{1/2}} e^{-(x_0 - x)^2/4t} \delta'(x_0) dx_0$$
$$= \int_{-\infty}^{\infty} \frac{1}{[4\pi t]^{1/2}} \frac{d}{dx_0} \left[ e^{-(x_0 - x)^2/4t} \right] \delta(x_0) dx_0$$
$$= \frac{x}{4\pi^{1/2}t^{3/2}} e^{-x^2/4t} = -\frac{\partial}{\partial x} \left[ \frac{e^{-x^2/4t}}{(4\pi t)^{1/2}} \right] = -\frac{\partial}{\partial x} C(x, t | 0, 0). \quad (7.17)$$

Now $C(x, t | 0, 0)$ is a solution of the homogeneous heat equation for $t > 0$ and therefore any space or time derivative is also a solution, since the heat equation has constant coefficients. Therefore $u(x, t)$ satisfies the homogeneous heat equation for $t > 0$, as could also be verified by straightforward differentiation. For each fixed $x$, we have

$$\lim_{t \to 0^+} u(x, t) = 0.$$ 

At first this result is shocking, for one would expect that the only solution of

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \quad -\infty < x < \infty, \quad t > 0; \quad u(x, 0) = 0, \quad (7.18)$$

should be $u(x, t) \equiv 0$. Physically this must be true if we have a rod whose initial temperature is 0 and is subject to no sources for $t > 0$. The paradox is resolved if we examine more closely the statement that $u(x, 0) = 0$. It is not enough to require that for each fixed $x$, $\lim_{t \to 0^+} u(x, t) = 0$, for then (7.17) would still satisfy (7.18) yet does not correspond to a zero initial temperature but rather to an initial dipole. We observe that (7.17) becomes positively infinite as $t \to 0^+$ along the $xt$ curve $x = 2\sqrt{t}$ and becomes negatively infinite as $t \to 0^+$ along the $xt$ curve $x = -2\sqrt{t}$. Thus the function defined by (7.17) for $t > 0$ and identically 0 for $t = 0$ is not continuous in the closed region $-\infty < x < \infty, \quad t \geq 0$, in spite of the fact that for each $x$, $\lim_{t \to 0^+} u(x, t) = 0$. Let us calculate the distributional limit of (7.17) as $t \to 0^+$. Let $\phi(x)$ be a test function; we want to find

$$\lim_{t \to 0^+} \int_{-\infty}^{\infty} u(x, t)\phi(x) dx = \lim_{t \to 0^+} \int_{-\infty}^{\infty} \frac{x}{4\pi^{1/2}t^{3/2}} e^{-x^2/4t} \phi(x) dx.$$

For each $t > 0$, by integration by parts, we have

$$\int_{-\infty}^{\infty} \frac{x}{4\pi^{1/2}t^{3/2}} e^{-x^2/4t} \phi(x) dx = \int_{-\infty}^{\infty} \frac{e^{-x^2/4t}}{[4\pi t]^{1/2}} \phi'(x) dx,$$
and therefore
\[
\lim_{t \to 0^+} \int_{-\infty}^{\infty} u(x, t)\varphi(x)dx = \varphi'(0),
\]
or
\[
\lim_{t \to 0^+} u(x, t) = -\delta'(x),
\]
which is, of course, the correct answer.

Let us now return to (7.15) and its solution (7.16). We can transform (7.16) into an expression better suited for calculations when \( t \) is small. Let
\[
z = \frac{x_0 - x}{2^{1/2}t},
\]
so that
\[
u(x, t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x + 2z\sqrt{t})e^{-z^2} dz,
\]
which is an alternative form for (7.16) and which makes it almost transparent that the initial value \( f(x) \) is taken on as \( t \to 0 \).

If \( f(x) \) can be expanded in a Taylor series, we have,
\[
f(x + 2z\sqrt{t}) = f(x) + 2z\sqrt{t}f'(x) + 2z^2tf''(x) + \cdots,
\]
and, substituting in (7.19),
\[
u(x, t) = f(x) + tf''(x) + \cdots.
\]
There is no guarantee that this series converges, but its first two terms may provide a reasonable approximation to \( u(x, t) \) for small \( t \). In fact, if \( f(x) \) is a quadratic polynomial \( Ax^2 + Bx + C \), then the two terms retained in (7.20) yield the exact solution \( Ax^2 + Bx + C + 2At \).

The solution (7.16) was derived by using the causal fundamental solution, which, in turn, was found by applying a Fourier transform on the space coordinate \( x \). We could just as well have taken the Fourier transform of (7.15) directly and used the convolution theorem to get (7.16). A different approach consists of taking the Laplace transform of (7.13) on the time coordinate. This method has the advantage of giving us some flexibility in inverting the transform so as to obtain computationally useful forms for \( u(x, t) \) when \( t \) is small or large. We shall illustrate this technique at a later stage when dealing with more difficult examples.

**EXERCISES**

7.1 When dealing with the problem
\[
\frac{\partial u}{\partial t} - \nabla^2 u = q(x, t), \quad x \in R, \quad t > 0; \quad u(x, 0) = f(x),
\]
\[
\frac{\partial u}{\partial n} (x, t) = h(x, t), \quad x \text{ on } \sigma, \quad t > 0,
\]
we need a causal fundamental solution which satisfies \( \partial g/\partial n = 0 \) when \( x \) is on \( \sigma \). By the method used in deriving (7.14), show that

\[
u(x, t) = \int_{0}^{t} dt_{0} \int_{R} d\sigma \, g(x, t | x_{0}, t_{0}) q(x_{0}, t_{0}) + \int_{R} d\sigma \, g(x, t | x_{0}, 0) f(x_{0}) + \int_{0}^{t} dt_{0} \int_{\sigma} dS_{0} \, g(x, t | x_{0}, t_{0}) h(x_{0}, t_{0}).
\]

7.2 Express the solution of

\[
\frac{\partial u}{\partial t} - \nabla^{2}u = q(x, t), \quad x \text{ in } R, \quad t > 0; \quad u(x, 0) = f(x);
\]

\[
\frac{\partial u}{\partial n} + \theta u = h(x, t), \quad x \text{ on } \sigma, \quad t > 0
\]

in terms of an appropriate causal fundamental solution. Here \( \theta \) is a given positive constant.

7.3 Equivalence of initial and boundary data to additional source terms. The solution \( u(x, t) \) of (7.3) is defined only for \( x \) in \( R, t > 0 \). Let us extend \( u \) to the whole of space-time by requiring that the function vanish outside the semi-infinite cylinder in Figure 7.1. This extended function will be called \( v(x, t) \) and we can write

\[
v(x, t) = H(t)H_{R}(x)u(x, t),
\]

where \( H(t) \) is the usual Heaviside function of \( t \) and \( H_{R}(x) \) is 1 when \( x \) is in \( R \) and 0 otherwise. The function \( v(x, t) \) obviously has certain discontinuities on the boundary of the semi-infinite cylinder of Figure 7.1; it will satisfy the heat equation with additional source terms concentrated on this boundary. We proceed with some formal calculations:

\[
\frac{\partial v}{\partial t} = H(t)H_{R}(x) \frac{\partial u}{\partial t} + \delta(t)H_{R}(x)u(x, t)
\]

\[
= H(t)H_{R}(x) \frac{\partial u}{\partial t} + H_{R}(x)u(x, 0)\delta(t),
\]

\[
\nabla^{2}v = H(t)\nabla^{2}(H_{R}u) = H(t)\left\{ H_{R} \nabla^{2}u - \frac{\partial u}{\partial n} \bigg|_{\sigma} \delta_{\sigma}(x) - u \bigg|_{\sigma} \frac{\partial \delta_{\sigma}}{\partial n} \right\},
\]

the last result being a consequence of (5.16). We therefore have

\[
\frac{\partial v}{\partial t} - \nabla^{2}v = H(t)H_{R}(x)q(x, t) + H_{R}(x)f(x)\delta(t)
\]

\[
+ H(t)h(x, t) \bigg|_{\sigma} \frac{\partial \delta_{\sigma}(x)}{\partial n} + H(t) \frac{\partial u}{\partial n} \bigg|_{\sigma} \delta_{\sigma}(x).
\]
All terms on the right side are known except the last one. The solution of (7.21) is defined only up to a solution of the homogeneous equation. By requiring \( v \) to vanish for \( t < 0 \) we obtain the solution we want. Now let \( g(x, t \mid x_0, t_0) \) be any causal fundamental solution for the heat equation (not necessarily satisfying any particular boundary conditions on \( \sigma \)). By the basic property of a fundamental solution, one solution of

\[
\frac{\partial v}{\partial t} - \nabla^2 v = p(x, t)
\]

is given by

\[
v(x, t) = \int_{R_n} dx_0 \int_{-\infty}^{t} dt_0 g(x, t \mid x_0, t_0)p(x_0, t_0).
\]

Since \( g \equiv 0 \) for \( t < t_0 \), this reduces to

\[
v = \int_{R_n} dx_0 \int_{-\infty}^{t} dt_0 g(x, t \mid x_0, t_0)p(x_0, t_0),
\]

where \( p \) is the right side of (7.21). Now \( p(x_0, t_0) = 0 \) for \( t_0 < 0 \), so that \( v(x, t) = 0 \) for \( t < 0 \). Hence we are finding the desired solution of (7.21). For \( t > 0 \) and \( x \in R \), we have

\[
v(x, t) = \int_{R} dx_0 \int_{0}^{t} dt_0 g(x, t \mid x_0, t_0)q(x_0, t_0) + \int_{R} g(x, t \mid x_0, 0)f(x_0)dx_0
\]

\[+ \int_{\sigma} dS_0 \int_{0}^{t} dt_0 g(x, t \mid x_0, t_0) \frac{\partial u}{\partial n_0} (x_0)\]

\[- \int_{\sigma} dS_0 \int_{0}^{t} dt_0 h(x_0, t_0) \frac{\partial g}{\partial n_0} (x, t \mid x_0, t_0).\]

If \( g \) is chosen to be the causal fundamental solution for \( R \), then \( g(x, t \mid x_0, t_0) \) vanishes for \( x \) on \( \sigma \) and the third integral is 0. We then obtain the previous result (7.14).

### 7.3 METHODS FOR FINDING THE CAUSAL GREEN'S FUNCTION

The solution of (7.8) is usually obtained by either a Laplace transform on time or by an expansion in spatial eigenfunctions, but in some special cases the method of images can be used. Consider, for instance, a semiinfinite rod, \( 0 < x < \infty \). Then \( g(x, t \mid x_0, 0) \) satisfies

\[
\left( \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \right) g(x, t \mid x_0, 0) = \delta(x - x_0)\delta(t), \quad 0 < x, x_0 < \infty, \quad -\infty < t < \infty;
\]

\[
g(x, t \mid x_0, 0) = 0, \quad t < 0, \quad 0 < x, x_0 < \infty; \tag{7.22}
\]

\[
g(0, t \mid x_0, 0) = 0, \quad -\infty < t < \infty, \quad 0 < x_0 < \infty.
\]
Let us replace our problem by one for an infinite rod with a unit positive source at \((x_0, 0)\) and a unit negative source at \((-x_0, 0)\). The temperature for the new problem is

\[
  w = C(x, t \mid x_0, 0) - C(x, t \mid -x_0, 0),
\]

where \(C\) is given by (7.10) with \(n = 1\). Therefore,

\[
  w = \frac{H(t)}{(4\pi t)^{1/2}} \left[ e^{-(x-x_0)^2/4t} - e^{-(x+x_0)^2/4t} \right].
\]

The function \(w\) satisfies (7.22) since the image source at \((-x_0, 0)\) is outside the original semi-infinite rod and clearly \(w\) vanishes for \(x = 0\). Assuming uniqueness, \(w\) is the required solution of (7.22). A time translation gives

\[
  g(x, t \mid x_0, t_0) = \frac{H(t - t_0)}{[4\pi(t-t_0)]^{1/2}} \left[ e^{-(x-x_0)^2/4(t-t_0)} - e^{-(x+x_0)^2/4(t-t_0)} \right],
\]

\[
  0 < x, x_0 < \infty. \quad (7.23)
\]

The solution of the initial value problem

\[
  \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \quad x > 0, \quad t > 0; \quad u(0, t) = 0, \quad u(x, 0) = f(x), \quad (7.24)
\]

is then obtained from (7.14) and (7.23) as

\[
  u(x, t) = \int_0^\infty \frac{1}{[4\pi t]^{1/2}} \left[ e^{-(x-x_0)^2/4t} - e^{-(x+x_0)^2/4t} \right] f(x_0) dx_0. \quad (7.25)
\]

By using the new variables

\[
  z = \frac{x_0 - x}{2t^{1/2}} \quad \text{and} \quad \frac{x_0 + x}{2t^{1/2}}
\]

in the first and second integrals, respectively, we obtain

\[
  u(x, t) = \frac{1}{\sqrt{\pi}} \int_{-x/2\sqrt{t}}^{x_0} e^{-z^2} f(x + 2z\sqrt{t}) dz
  - \frac{1}{\sqrt{\pi}} \int_{x/2\sqrt{t}}^{x_0} e^{-z^2} f(-x + 2z\sqrt{t}) dz. \quad (7.26)
\]

As \(t \to 0^+\), the second integral approaches 0 and the first approaches \(f(x)\), thereby showing that the initial value is assumed. In the particular case \(f = 1\), we find

\[
  u(x, t) = \frac{2}{\sqrt{\pi}} \int_0^{x/2\sqrt{t}} e^{-z^2} dz = \text{erf} \left( \frac{x}{2\sqrt{t}} \right), \quad (7.27)
\]

where \(\text{erf}\) is the usual error function.
If $x/2\sqrt{t}$ is small, we can expand erf in a power series as follows:

$$e^{-z^2} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{n!}$$

$$\int_0^y e^{-z^2} \, dz = \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n+1}}{(n!)(2n+1)}.$$  

Thus

$$u(x, t) \sim \frac{2}{\sqrt{\pi}} \left[ \frac{x}{2\sqrt{t}} - \frac{1}{3} \left( \frac{x}{2\sqrt{t}} \right)^3 + \cdots \right], \quad (7.28)$$

the first two terms of which yield a good approximation if $x/2\sqrt{t}$ is small; thus for a fixed $x > 0$, the representation is good for large $t$. The expression (7.28) was obtained from the exact solution (7.27). We now show how (7.28) could be derived without appealing directly to the exact solution. Consider (7.24) with $f(x) = 1$. Multiply the differential equation by $e^{-st}$ and integrate from $t = 0$ to $t = \infty$ to find

$$-\frac{d^2 \tilde{u}}{dx^2} + st \tilde{u} = 1, \quad 0 < x < \infty, \quad \tilde{u}(0, s) = 0, \quad (7.29)$$

where $\tilde{u}(x, s)$ is the Laplace transform of $u(x, t)$; that is,

$$\tilde{u}(x, s) = \int_0^\infty e^{-st} u(x, t) \, dt.$$ 

The solution of (7.29), which is finite at $x = +\infty$ for $\Re s > 0$, is

$$\tilde{u}(x, s) = \frac{1}{s} \left[ 1 - e^{-x\sqrt{s}} \right]. \quad (7.30)$$

Using the asymptotic formula for inverting Laplace transforms for large $t$ [see Appendix B, equation (B.10)], we again obtain (7.28).

If we are interested in small values of $t$ instead of large ones, we rewrite (7.27) as

$$u(x, t) = 1 - \frac{2}{\sqrt{\pi}} \int_{x/2\sqrt{t}}^\infty e^{-z^2} \, dz, \quad (7.31)$$

where $x/2\sqrt{t}$ is large. Now

$$\int_A^\infty e^{-z^2} \, dz = -\frac{1}{2z^2} (2ze^{-z^2}) \, dz,$$

which, using integration by parts, reduces to

$$\frac{1}{2A} e^{-A^2} - \int_A^\infty \frac{e^{-z^2}}{2z^2} \, dz. \quad (7.32)$$
If $A$ is large, the last integral is much smaller than
\[
\int_{A}^{\infty} e^{-x^2} \, dx,
\]
so that, with $A = x/2\sqrt{t}$,
\[
u(x, t) \sim 1 - \frac{2\sqrt{t} e^{-x^2/4t}}{\sqrt{\pi x}}, \quad \text{for } x > 0, \quad t \text{ small.}
\]

Next, we turn to the boundary value problem for the semi-infinite rod:
\[
\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < x < \infty, \quad t > 0; \quad u(0, t) = h(t), \quad t > 0; \quad u(x, 0) = 0, \quad x > 0.
\]

The boundary $\sigma$ consists of the single point $x = 0$ and
\[
\frac{\partial g(x, t \mid x_0, t_0)}{\partial n_0} = -\frac{\partial}{\partial x_0} g(x, t \mid x_0, t_0) \bigg|_{x_0 = 0} = -\frac{H(t - t_0)xe^{-x^2/4(t-t_0)}}{(4\pi)^{1/2}(t-t_0)^{3/2}},
\]
so that, from (7.14),
\[
u(x, t) = \int_{0}^{t} \frac{xe^{-x^2/4(t-t_0)}}{(4\pi)^{1/2}(t-t_0)^{3/2}} h(t_0) dt_0, \quad x > 0, \quad t > 0.
\]

One easily verifies that (7.35) satisfies the differential equation and initial condition in (7.34), but it is somewhat harder to show that the boundary condition at $x = 0$ is satisfied. The expression (7.35) vanishes for $x = 0$, which seems to contradict the requirement that the boundary temperature be $h(t)$; of course, this boundary condition means that $\lim_{x \to 0^+} u(x, t) = h(t)$ in some sense, and we now show that (7.35) does in fact have the required limiting value.

For each $\varepsilon > 0$, consider the sequence of nonnegative functions of the real variable $z$,
\[
s_{\varepsilon}(z) = \begin{cases} 
\varepsilon \exp\left(-\varepsilon^2/4z\right)/(4\pi)^{1/2}z^{3/2}, & z > 0; \\
0, & z < 0.
\end{cases}
\]

We claim that $s_{\varepsilon}(z)$ is a $\delta$ sequence. It suffices to show that
\[
\lim_{\varepsilon \to 0} \int_{A}^{B} s_{\varepsilon}(z) \, dz = \begin{cases} 
0, & \text{if } 0 < A < B; \\
1, & \text{if } A = 0 \text{ and } B > 0.
\end{cases}
\]

Making the change of variables $w = \varepsilon / 2z^{1/2}$, we find
\[
\int_{A}^{B} s_{\varepsilon}(z) \, dz = \frac{2}{\pi^{1/2}} \int_{\varepsilon / 2\sqrt{A}}^{\varepsilon / 2\sqrt{B}} e^{-w^2} \, dw,
\]
where the upper limit is to be understood as $\infty$ if $A = 0$. Therefore (7.36) follows, and $s_{\varepsilon}(z)$ is a $\delta$ sequence. Thus

$$\lim_{\varepsilon \to 0+} \int_{0}^{A} s_{\varepsilon}(z) \varphi(z) dz = \varphi(0),$$

and, setting $A = t$, $\varepsilon = x$, $z = t - t_0$ (where $t_0$ is a new variable of integration), and $\varphi(t - t_0) = h(t_0)$, we find

$$\lim_{x \to 0+} \int_{0}^{t} \frac{x h(t_0) e^{-x^2/(4(t-t_0))}}{(4\pi)^{1/2} (t-t_0)^{3/2}} \, dt_0 = h(t),$$

which is the desired result.

Thus we have verified that (7.35) satisfies (7.34). However, the expression (7.35) is discontinuous at $x = 0$ and is unsatisfactory near $x = 0$ for computational purposes. The following trick can be used to circumvent this difficulty. Let $k(x, t)$ be any function which takes on the prescribed boundary value $h(t)$ at $x = 0$ and define

$$v(x, t) = u(x, t) - k(x, t),$$

which satisfies

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = - \frac{\partial k}{\partial t} + \frac{\partial^2 k}{\partial x^2}, \quad x > 0, \quad t > 0; \quad v(0, t) = 0, \quad t > 0; \quad v(x, 0) = -k(x, 0), \quad x > 0. \quad (7.37)$$

The solution of (7.37) can be found by using (7.14) and will have the property $\lim_{x \to 0+} v(x, t) = 0$, where both terms which make up $v(x, t)$ vanish at $x = 0$. Then

$$u(x, t) = k(x, t) + v(x, t)$$

is more convenient for calculation near $x = 0$. The method is particularly useful if $k(x, t)$ can be chosen to satisfy the homogeneous heat equation.

As an example, consider (7.34) with $h(t) = 1$ and let $k(x, t) = 1$; then (7.37) becomes the initial value problem

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = 0, \quad x > 0, \quad t > 0; \quad v(0, t) = 0, \quad t > 0; \quad v(x, 0) = -1, \quad x > 0,$$

whose solution, from (7.24) and (7.27), is

$$v(x, t) = -\text{erf} \frac{x}{2\sqrt{t}}.$$

Thus the solution of (7.34) with $h(t) = 1$ is

$$u(x, t) = 1 - \text{erf} \frac{x}{2\sqrt{t}}, \quad (7.38)$$

which clearly satisfies both the boundary and initial conditions.
The method of images can also be used to construct the causal fundamental solution for a semiinfinite rod when the left end of the rod is insulated. Thus the solution of

$$\frac{\partial g}{\partial t} - \frac{\partial^2 g}{\partial x^2} = \delta(x - x_0)\delta(t), \quad 0 < x, x_0 < \infty, \quad -\infty < t < \infty,$$

$$g(x, t | x_0, 0) = 0, \quad t < 0$$

$$\frac{\partial g}{\partial x} (x, t | x_0, 0) \bigg|_{x = 0} = 0, \quad x_0 > 0,$$

is found by placing an image source at $x = -x_0$, but this time we use a positive source to satisfy the new boundary condition. This gives

$$g(x, t | x_0, 0) = \frac{1}{[4\pi t]^{1/2}} \left[ e^{-(x-x_0)^2/4t} + e^{-(x+x_0)^2/4t} \right], \quad t > 0; \quad x_0, x > 0.$$

(7.39)

If the boundary condition at $x = 0$ is the radiative one,

$$\frac{\partial g}{\partial n} + \theta g = 0,$$

that is,

$$\frac{\partial g}{\partial x} - \theta g = 0, \quad x = 0,$$

the straightforward method of images fails. The following trick works in this case. As we know, $g(x, t | x_0, 0)$ satisfies the initial value problem

$$\frac{\partial g}{\partial t} - \frac{\partial^2 g}{\partial x^2} = 0, \quad t > 0; \quad x_0, x > 0; \quad g(x, 0 | x_0, 0) = \delta(x - x_0);$$

$$\frac{\partial g}{\partial x} - \theta g = 0, \quad x = 0.$$

(7.40)

Let

$$v = \frac{\partial g}{\partial x} - \theta g;$$

then $v$ satisfies

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = 0, \quad t > 0, \quad x_0, x > 0;$$

$$v(x, 0 | x_0, 0) = \delta'(x - x_0) - \theta \delta(x - x_0), \quad v = 0 \text{ at } x = 0.$$

(7.41)
From (7.24) and (7.25), we have

\[
v(x, t) = \int_{0}^{\infty} \frac{1}{(4\pi t)^{1/2}} \left[ e^{-(x-\xi)^2/4t} - e^{-(x+\xi)^2/4t} \right] \left[ \delta'(\xi - x_0) - \theta \delta(\xi - x_0) \right] d\xi
\]

\[
= -\theta C(x, t \mid x_0, 0) + \frac{d}{dx} C(x, t \mid x_0, 0)
\]

\[
+ \theta C(x, t \mid -x_0, 0) + \frac{d}{dx} C(x, t \mid -x_0, 0),
\]

where \( C \) is the causal fundamental solution for free space,

\[
C(x, t \mid x_0, 0) = \frac{1}{(4\pi t)^{1/2}} e^{-(x-x_0)^2/4t}.
\]

Since \( v = \frac{dg}{dx} - \theta g \), and \( v \) is required to vanish at \( x = +\infty \), we find

\[
g(x, t \mid x_0, 0) = -e^{\theta x} \int_{x}^{\infty} e^{-\theta \xi} v(\xi, t) d\xi
\]

\[
= -e^{\theta x} \int_{x}^{\infty} e^{-\theta \xi} \left[ -\theta C(\xi, t \mid x_0, 0) + \frac{d}{d\xi} C(\xi, t \mid x_0, 0) \right] d\xi
\]

\[
- e^{\theta x} \int_{x}^{\infty} e^{-\theta \xi} \left[ \theta C(\xi, t \mid -x_0, 0) + \frac{d}{d\xi} C(\xi, t \mid -x_0, 0) \right] d\xi.
\]

Integrating by parts the terms with \( d/d\xi \), we obtain

\[
g(x, t \mid x_0, 0) = C(x, t \mid x_0, 0) + C(x, t \mid -x_0, 0)
\]

\[
- 2\theta e^{\theta x} \int_{x}^{\infty} e^{-\theta \xi} C(\xi, t \mid -x_0, 0) d\xi,
\]

which, after completing the square in the remaining integral, becomes

\[
g(x, t \mid x_0, 0) = \frac{1}{(4\pi t)^{1/2}} \left[ e^{-(x-x_0)^2/4t} + e^{-(x+x_0)^2/4t} \right]
\]

\[
- \theta e^{\theta x} \sigma_{x_0} \left[ 1 - \text{erf} \left( \frac{x + x_0}{2\sqrt{t}} + \theta \sqrt{t} \right) \right].
\]

(7.43)

By setting \( z = x - x \) in the integral in (7.42), we can also obtain the form

\[
g(x, t \mid x_0, 0) = \frac{1}{(4\pi t)^{1/2}} \left[ e^{-(x-x_0)^2/4t} + e^{-(x+x_0)^2/4t} \right]
\]

\[
- 2\theta \int_{0}^{\infty} e^{-\theta z} e^{-(x+x_0+z)^2/4t} dz.
\]

(7.44)

In the next section we show how to obtain this result in a less circuitous manner.
Next we turn to problems for bounded regions. Consider first heat conduction in an insulated thin ring of constant cross section and of perimeter 1. Such a ring can be built by taking a thin rod of length 1 and bending it so that the ends meet perfectly. We identify points along the center line of the ring by a linear coordinate $x$ ranging from $-\frac{1}{2}$ to $\frac{1}{2}$ (see Figure 7.3). The ring is at zero temperature until time 0 when an instantaneous unit source of heat is liberated at $x = 0$. Because the ring is insulated, the heat flow is one-dimensional along the center line of the ring. The only novel feature of the problem is the nature of the boundary conditions; since the points $x = -\frac{1}{2}$ and $x = +\frac{1}{2}$ represent the same point on the ring, we must have

$$\left. \frac{\partial^{(k)} u}{\partial x^{(k)}} \right|_{x=1/2} = \left. \frac{\partial^{(k)} u}{\partial x^{(k)}} \right|_{x=-1/2}, \quad k = 0, 1, 2, \ldots,$$

where $u$ is the temperature in the ring. It suffices to require the above continuity conditions for $k = 0$ and $k = 1$, the others following by differentiation of the differential equation.

Thus we wish to find the causal fundamental solution of

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \delta(x)\delta(t); \quad -\frac{1}{2} < x < \frac{1}{2}, \quad -\infty < t < \infty;$$

$$u \left( -\frac{1}{2}, t \right) = u \left( \frac{1}{2}, t \right); \quad \frac{\partial u}{\partial x} \left( -\frac{1}{2}, t \right) = \frac{\partial u}{\partial x} \left( \frac{1}{2}, t \right). \quad (7.45)$$

The solution of (7.45) can be obtained by the method of images. Consider an infinite rod with positive unit sources located at all integer values of $x$. Clearly the resulting temperature is periodic with period 1, so that the boundary conditions in (7.45) are satisfied; moreover, the part of the rod between $x = -\frac{1}{2}$ and $x = \frac{1}{2}$ contains only the single source at $x = 0$. If we believe in uniqueness of the solution of (7.45), this solution must coincide
with the temperature between \(-\frac{1}{2}\) and \(\frac{1}{2}\) in the infinite rod. But this latter temperature is

\[ \theta(x, t) = \frac{1}{(4\pi t)^{1/2}} \sum_{n=-\infty}^{\infty} e^{-(x-n)^2/4t}, \quad (7.46) \]

which therefore coincides with the solution of (7.45) when \(-\frac{1}{2} < x < \frac{1}{2}\). The function \(\theta(x, t)\) is related to the Jacobi theta function, but we shall not try here to clarify the bewildering variety of notations found in the literature.

We can obtain another form for (7.46) by using the Poisson sum formula (1.23b) of Chapter 1. We repeat the formula for convenience; if \(v^\wedge(\omega)\) is the Fourier transform of \(v(x)\), then

\[ \sum_{n=-\infty}^{\infty} v(n) = \sum_{n=-\infty}^{\infty} v^\wedge(2n\pi). \quad (7.47) \]

Setting \(v(x) = (4\pi t)^{-1/2} e^{-(x-a)^2/4t}\), an easy calculation shows that

\[ v^\wedge(\omega) = e^{ix\omega} e^{-\omega^2}. \]

Substituting in the Poisson formula, we find

\[ \theta(x, t) = \sum_{n=-\infty}^{\infty} e^{i2n\pi x} e^{-4n^2\pi^2 t} = 1 + 2 \sum_{n=1}^{\infty} \cos 2n\pi x e^{-4n^2\pi^2 t}, \quad (7.48) \]

which should be compared with (7.46). The formula (7.46) is useful computationally for \(t\) small and (7.48) for \(t\) large. The form (7.48) can also be obtained by an eigenfunction expansion (see Exercise 7.4).

Let us now consider the causal solution for a rod of length \(l\) whose ends \(x = 0\) and \(x = l\) are kept at zero temperature. The source is introduced at the point \(x = \xi\) at time \(t = 0\). Again we consider an auxiliary infinite rod (see Figure 7.4) and to satisfy the boundary conditions at \(x = 0\) and \(x = l\) we must use the array of sources:

\[
\begin{align*}
&\xi - 2l & -\xi & \xi & -\xi + 2l & \xi + 2l \\
&+ & - & + & - & +
\end{align*}
\]

\[ x = 0 \quad x = l \]

\textbf{FIGURE 7.4}

Figure 7.4) and to satisfy the boundary conditions at \(x = 0\) and \(x = l\) we must use the array of sources:

\[
\begin{align*}
1 \text{ at } x &= \xi + 2nl, \quad n = \ldots, -2, -1, 0, 1, 2, \ldots; \\
-1 \text{ at } x &= -\xi + 2nl, \quad n = \ldots, -2, -1, 0, 1, 2, \ldots.
\end{align*}
\]

The resulting temperature between \(x = 0\) and \(x = l\) will satisfy the same boundary value problem as the one for the finite rod with the ends \(x = 0\)
and \( x = l \) kept at zero temperature. Therefore, the causal Green’s function is given by
\[
g(x, t | \xi, 0) = \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} \left[ e^{-(x-\xi-2nl)^2/4t} - e^{-(x+\xi-2nl)^2/4t} \right]
\]
\[
= \frac{1}{2l} \left[ \theta \left( \frac{x-\xi}{2l}, \frac{t}{4l^2} \right) - \theta \left( \frac{x+\xi}{2l}, \frac{t}{4l^2} \right) \right].
\] (7.49)

From (7.48), we find, after some simple manipulations,
\[
g(x, t | \xi, 0) = \frac{2}{l} \sum_{n=1}^{\infty} \sin \frac{n\pi x}{l} \sin \frac{n\pi \xi}{l} e^{-n^2\pi^2t/l^2},
\] (7.50)
a result which will also be obtained later by expanding in the space eigenfunctions.

**Expansion in Space Eigenfunctions**

Consider the initial value problem for the heat equation when no sources are present and the boundary temperature is zero. Thus we wish to solve the problem
\[
\frac{\partial u}{\partial t} - \nabla^2 u = 0, \quad x \text{ in } R, \quad t > 0;
\]
\[
u(x, 0) = f(x), \quad x \text{ in } R,
\]
\[
u(x, t) = 0, \quad x \text{ on } \sigma, \quad t > 0.
\] (7.51)

We first try to find solutions of the differential equation as products of functions of time and space; we require these separable solutions to satisfy the boundary condition but not the initial condition. Then by superposition we hope to also satisfy the initial condition. Let
\[
u^*(x, t) = X(x)T(t)
\]
be a separable solution satisfying the boundary condition; substituting in the differential equation, we find, by the usual arguments,
\[
-\nabla^2 X = \lambda X, \quad x \text{ in } R; \quad X(x) = 0, \quad x \text{ on } \sigma,
\] (7.52)
\[
T' = -\lambda T,
\] (7.53)
where \( \lambda \) is a separation constant. The eigenvalue problem (7.52) for the negative Laplacian was discussed in Section 6.6. If \( R \) is a bounded region, there is a denumerable sequence of positive eigenvalues
\[
\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n \leq \cdots,
\]
which are listed in increasing order with due regard to multiplicity. The corresponding eigenfunctions
\[ \varphi_1(x), \varphi_2(x), \ldots \]
can be chosen to form a complete orthonormal set.

Thus any function of the form
\[ \varphi_i(x)e^{-\lambda_it} \]
is a solution of the differential equation and satisfies the boundary condition. Setting aside any questions regarding convergence and term-by-term differentiability, the same is true for
\[ \sum_{i=1}^{\infty} c_i \varphi_i(x)e^{-\lambda_it}, \]  \hspace{1cm} (7.54)
where the \( \{c_i\} \) are constants which are to be adjusted so as to satisfy the initial condition
\[ \sum_{i=1}^{\infty} c_i \varphi_i(x) = f(x). \]

This equation will hold if and only if
\[ c_i = \langle f, \varphi_i \rangle = \int_{\mathbb{R}} f(x)\overline{\varphi_i(x)}dx. \]  \hspace{1cm} (7.55)

We have therefore found the formal solution of (7.51),
\[ u(x, t) = \sum_{i=1}^{\infty} \langle f, \varphi_i \rangle \varphi_i(x)e^{-\lambda_it}. \]  \hspace{1cm} (7.56)

Since all \( \{\lambda_i\} \) are positive, the temperature in \( \mathbb{R} \) approaches 0 as \( t \to \infty \), as expected. For the causal Green's function \( g(x, t | x_0, 0) \), we have \( f(x) = \delta(x - x_0) \), so that
\[ g(x, t | x_0, 0) = \sum_{i=1}^{\infty} \varphi_i(x)\overline{\varphi_i(x_0)}e^{-\lambda_it}. \]
and, by a time translation,
\[ g(x, t | x_0, t_0) = \sum_{i=1}^{\infty} \varphi_i(x)\overline{\varphi_i(x_0)}e^{-\lambda_i(t-t_0)}. \]  \hspace{1cm} (7.57)

If we want to solve (7.51) for a different homogeneous boundary condition, all we have to do is to consider (7.52) for the new boundary condition. For instance, if \( \partial u/\partial n = 0 \) on \( \sigma \), that is, if the boundary is insulated, then (7.52) becomes
\[ -\nabla^2 X = \lambda X, \quad x \text{ in } \mathbb{R}; \quad \frac{\partial X}{\partial n} = 0, \quad x \text{ on } \sigma. \]
In this case the eigenvalues are nonnegative and zero is an eigenvalue to which corresponds an eigenfunction constant in $R$. Denoting the eigenvalues in increasing order by $\{\mu_i\}_{i=0}^{\infty}$ and the orthonormal eigenfunctions by $\{\psi_i\}$, we find

$$u(x, t) = \sum_{i=0}^{\infty} \langle f, \psi_i \rangle \psi_i(x)e^{-\mu_i t}.$$  

As $t \to \infty$, all terms, except the one corresponding to $i = 0$, vanish. Thus

$$\lim_{t \to \infty} u(x, t) = \psi_0(x) \int_R f \psi_0 \ dx,$$

and since

$$\psi_0(x) = \frac{1}{V^{1/2}},$$

where $V$ is the volume of $R$, we find

$$\lim_{t \to \infty} u(x, t) = \frac{1}{V} \int_R f(x) dx.$$

Thus the temperature approaches a constant value, the average of the initial temperature. This is, of course, just what is expected on physical grounds when the boundary is insulated.

If the region $R$ is unbounded, the problem (7.52) does not lead to a discrete spectrum. Instead $\lambda$ can take on any of a continuous set of values and the eigenfunctions are improper (not square-integrable on $R$). The sum (7.54) becomes an integral and we must use an integral transformation to solve (7.51). We shall discuss individual examples on an ad hoc basis rather than to try to develop a general theory.

Returning to the case of a bounded region, we show how an eigenfunction expansion based on (7.52) can be used to solve the general problem (7.3) even though the boundary conditions are inhomogeneous. The result can be obtained in two ways: either by a direct expansion in (7.3) or by using (7.57) in (7.14). The second method is simpler but, for variety, we adopt the first approach. The solution of (7.3) can, for any fixed $t$, be expanded in the complete set $\{\varphi_i(x)\}$ with coefficients depending parametrically on $t$. Thus

$$u(x, t) = \sum_{i=1}^{\infty} u_i(t) \varphi_i(x), \quad u_i(t) = \int_R u(x, t) \overline{\varphi_i(x)} dx. \quad (7.58)$$

Owing to the fact that $u(x, t)$ satisfies inhomogeneous boundary conditions whereas the $\{\varphi_i\}$ vanish on the boundary, the series (7.58) cannot converge uniformly. Therefore we do not dare differentiate the series term by term and we cannot substitute the series in (7.3). Instead, we multiply both sides of the differential equation by $\overline{\varphi_i(x)}$ and integrate over $R$, to obtain

$$\frac{d}{dt} u_i(t) - \int_R \overline{\varphi_i(x)} \nabla^2 u \ dx = \int_R \overline{\varphi_i(x)} q(x, t) dx.$$
The right side is \( q_i(t) \), the \( i \)th coefficient of the expansion of the known function \( q(x, t) \) in the orthonormal set \( \{ \varphi_i \} \). Since

\[
\int_R \bar{\varphi}_i(x) \nabla^2 u \, dx = \int_R u \nabla^2 \bar{\varphi}_i(x) \, dx + \int_\sigma \left( \bar{\varphi}_i \frac{\partial u}{\partial n} - u \frac{\partial \bar{\varphi}_i}{\partial n} \right) dS,
\]

\[
\nabla^2 \varphi_i = -\lambda_i \varphi_i = -\lambda_i \bar{\varphi}_i; \quad \bar{\varphi}_i(x) = 0, \quad x \text{ on } \sigma,
\]

we have

\[
\frac{du_i(t)}{dt} + \lambda_i u_i(t) = q_i(t) - \int_\sigma h(x, t) \frac{\partial \bar{\varphi}_i}{\partial n} \, dS. \tag{7.59}
\]

Moreover, from (7.58), we have the initial condition

\[
u_i(0) = \int_R f \bar{\varphi}_i \, dx = \langle f, \varphi_i \rangle = f_i.
\]

It is now a simple matter to solve the ordinary differential equation of the first order, (7.59), subject to the initial condition just mentioned. We find

\[
u_i(t) = f_i e^{-\lambda_i t} + \int_0^t e^{-\lambda_i (t-\tau)} \left[ q_i(\tau) - \int_\sigma h(x, \tau) \frac{\partial \bar{\varphi}_i}{\partial n} \, dS \right] d\tau.
\]

Substituting in (7.58),

\[
u(x, t) = \sum_{i=1}^\infty f_i e^{-\lambda_i t} \varphi_i(x) + \sum_{i=1}^\infty e^{-\lambda_i t} \varphi_i(x) \int_0^t e^{\lambda_i \tau} q_i(\tau) d\tau
\]

\[
- \sum_{i=1}^\infty e^{-\lambda_i t} \varphi_i(x) \int_0^t e^{\lambda_i \tau} h_i(\tau) d\tau,
\]

where

\[
h_i(\tau) = \int_\sigma h(x, \tau) \frac{\partial \bar{\varphi}_i}{\partial n} \, dS. \tag{7.60}
\]

**EXAMPLES**

**Example 1.** The Green’s function for a rod with ends at zero temperature. Then problem (7.52) reduces to the one-dimensional eigenvalue problem,

\[
- \frac{d^2 X}{dx^2} = \lambda X, \quad 0 < x < l; \quad X(0) = X(l) = 0,
\]

whose eigenvalues are

\[
\lambda_k = \frac{k^2\pi^2}{l^2}, \quad k = 1, 2, \ldots,
\]

with normalized eigenfunctions

\[
\varphi_k(x) = \left( \frac{2}{l} \right)^{1/2} \sin \frac{k\pi x}{l}.
\]
The Green's function for heat conduction can be calculated from (7.57):

\[ g(x, t \mid x_0, t_0) = \sum_{k=1}^{\infty} \frac{2}{l} \sin \frac{k\pi x}{l} \sin \frac{k\pi x_0}{l} e^{-k^2 \alpha^2 (t-t_0)/l^2}, \]

which is in agreement with (7.50).

**Example 2.** The semiinfinite rod, \(0 < x < \infty\), with radiation from the left end into a surrounding atmosphere at zero temperature. Then (7.52) becomes

\[ - \frac{d^2 X}{dx^2} = \lambda X, \quad 0 < x < \infty; \quad X'(0) - X(0) = 0. \quad (7.61) \]

By analogy with the treatment of such problems in Chapter 4, we expect a continuous spectrum. To obtain the spectral representation, we first construct the Green's function \(r(x \mid \xi; \lambda)\) for (7.61), that is, the Green's function for the space part of the heat-conduction problem. Whenever \(\lambda\) is not a nonnegative real number, the system

\[ - \frac{d^2 r(x \mid \xi; \lambda)}{dx^2} - \lambda r(x \mid \xi; \lambda) = \delta(x - \xi), \quad 0 < x, \xi < \infty; \]

\[ \left( \frac{dr}{dx} - r \right) \bigg|_{x=0} = 0, \quad \int_{0}^{\infty} |r|^2 \, dx < \infty \]

has a unique solution. In fact, an easy calculation shows

\[ r(x \mid \xi; \lambda) = \frac{i}{\nu(\nu + i)} [e^{i\nu x} (\sin \nu x + \nu \cos \nu x)], \]

where \(\nu\) is the square root of \(\lambda\) which has positive imaginary part and \(x_+ = \max (x, \xi), x_- = \min (x, \xi)\). On using the formula (4.44) (see also Exercise 4.24),

\[ \frac{1}{2\pi i} \int_{i \infty}^{0} r(x \mid \xi; \lambda) d\lambda = -\delta(x - \xi), \]

we find

\[ \delta(x - \xi) = \frac{2}{\pi} \int_{0}^{\infty} \frac{1}{\nu^2 + 1} [\sin \nu x + \nu \cos \nu x][\sin \nu \xi + \nu \cos \nu \xi] d\nu, \]

for \(0 < x, \xi < \infty\). This leads to the transform pair

\[ F(\nu) = \left( \frac{2}{\pi} \right)^{1/2} \int_{0}^{\infty} \frac{[\sin \nu x + \nu \cos \nu x]}{(\nu^2 + 1)^{1/2}} f(x) dx, \quad (7.62) \]

\[ f(x) = \left( \frac{2}{\pi} \right)^{1/2} \int_{0}^{\infty} \frac{[\sin \nu x + \nu \cos \nu x]}{(\nu^2 + 1)^{1/2}} F(\nu) d\nu, \quad 0 < x < \infty. \quad (7.63) \]

The second of these formulas may be regarded as the expansion of \(f(x)\) in terms of the continuous set of normalized pseudo eigenfunctions of (7.61), the
coefficient in the expansion being given by the first formula. To find the Green's function for the heat-conduction problem, we may proceed in either of two equivalent ways. We can multiply the heat equation by

$$
\left( \frac{2}{\pi} \right)^{1/2} \frac{\sin \nu x + \nu \cos \nu x}{(\nu^2 + 1)^{1/2}}
$$

(7.64)

and integrate over $x$ from 0 to $\infty$ to obtain an ordinary differential equation for the transform of the Green's function and then use (7.63) to find $g(x, t \mid \xi, 0)$. Alternatively, we use (7.57) directly with $\phi(x)$ defined by (7.64) and the summation replaced by an integration over the spectrum. This gives

$$
g(x, t \mid \xi, 0) = \frac{2}{\pi} \int_0^\infty \frac{1}{\nu^2 + 1} (\sin \nu x + \nu \cos \nu x)(\sin \nu \xi + \nu \cos \nu \xi)e^{-\nu^2 t} d\nu,
$$

(7.65)

which, after some manipulation can be reduced to the previously found result (7.44), with $x_0 = \xi$ and $\theta = 1$.

**Laplace Transform Method**

Consider again the general problem (7.3), repeated for convenience:

$$
\frac{\partial u}{\partial t} - \nabla^2 u = q(x, t), \quad x \text{ in } R, \quad t > 0;
$$

$$
u(x, 0) = f(x); \quad u(x, t) = h(x, t), \quad x \text{ on } \sigma, \quad t > 0.
$$

(7.66)

Let

$$
\tilde{u}(x, s) = \int_0^\infty e^{-st}u(x, t)dt,
$$

(7.67)

and multiply the differential equation by $e^{-st}$ and integrate from $t = 0$ to $t = \infty$. This yields

$$
-\nabla^2 \tilde{u}(x, s) + \int_0^\infty \frac{\partial u}{\partial t} e^{-st} dt = \int_0^\infty q(x, t)e^{-st} dt = \tilde{q}(x, s).
$$

If the first integral on the left is integrated by parts, one obtains

$$
-\nabla^2 \tilde{u}(x, s) + s\tilde{u}(x, s) = \tilde{q}(x, s) + f(x), \quad x \text{ in } R;
$$

$$
\tilde{u}(x, s) = \int_0^\infty h(x, t)e^{-st} dt = \tilde{h}(x, s), \quad x \text{ on } \sigma.
$$

(7.68)

This formal derivation can be justified if Re $s > 0$. To find $\tilde{u}$ we have to solve a problem in which only the space coordinates appear. Let $G(x \mid \xi; \lambda)$
be the Green’s function for the space problem
\[-\nabla^2 G - \lambda G = \delta(x - \xi), \quad x, \xi \text{ in } R;
\]
\[G(x \mid \xi; \lambda) = 0, \quad x \text{ on } \sigma, \quad \int_R |G|^2 \, dx < \infty.\]  
(7.69)

If \(\lambda\) is not in \([0, \infty)\), this problem has one and only one solution. In particular, if \(\lambda = -s\), where \(\text{Re}\, s > 0\), the solution always exists. Using this Green’s function, we can write the solution of (7.68) as
\[\tilde{u}(x, s) = \int_R G(x \mid \xi; -s)[\tilde{q}(\xi, s) + f(\xi)]d\xi - \int_\sigma \frac{\partial G(x \mid \xi; -s)}{\partial n} \tilde{h}(\xi, s)d\xi.\]  
(7.70)

It then remains to calculate \(u(x, t)\) by the Laplace inversion formula
\[u(x, t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \tilde{u}(x, s)e^{st}ds, \quad a > 0.\]  
(7.71)

Although the approach may look fairly complicated it has many advantages. If the data are simple enough \(\tilde{u}(x, s)\) can be calculated in closed form, possibly without recourse to (7.70). We may then be able to invert (7.71) exactly or, in more complicated problems, use various tricks for obtaining asymptotic formulas for small and large values of \(t\).

For the causal Green’s function of (7.66), our formulas simplify considerably. Then \(q = h = 0\), \(f(x) = \delta(x - x_0)\), and \(\tilde{u}(x, s) = G(x \mid x_0; -s)\). Consequently,
\[g(x, t \mid x_0, 0) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} G(x \mid x_0; -s)e^{st}ds, \quad a > 0.\]  
(7.72)

Although this is not the purpose of the Laplace transform procedure, it is gratifying that we can recover (7.57) from (7.72). In fact, by the bilinear formula for \(G(x \mid x_0; -s)\), we have
\[G(x \mid x_0; -s) = \sum_{k=1}^{\infty} \frac{\varphi_k(x)\varphi_k(x_0)}{s + \lambda_k}.\]

The integral in (7.72) can be closed to the left (see Figure 7.5) so as to include the simple poles of the integrand at \(s = -\lambda_k\). Using the theory of residues and disposing of the integrals over parts I, II, and III by the standard arguments, we find
\[g(x, t \mid x_0, 0) = 2\pi i \sum \text{residues of } \frac{e^{st}G(x \mid x_0; -s)}{2\pi i} = \sum_{k=1}^{\infty} e^{-\lambda_k t}\varphi_k(x)\varphi_k(x_0).\]

We emphasize once more that our goal in using the Laplace transform is not to obtain the last formula, which was already found by simpler means.
EXAMPLE

Consider the region $R$ exterior to the unit sphere in three dimensions. With $r$ the usual spherical coordinate, let us investigate the boundary value problem

$$\frac{\partial u}{\partial t} - \nabla^2 u = 0, \quad r > 1, \quad t > 0; \quad u|_{r=0} = 0; \quad u|_{r=1} = 1. $$

We can interpret the problem as follows: heat conduction in an infinite threedimensional medium with a spherical hole. The initial temperature is zero and the boundary is kept at temperature 1 for all $t > 0$. Clearly $u$ depends only on $r$ and $t$, so that we can write

$$\frac{\partial u}{\partial t} - \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial u}{\partial r} \right) = 0, \quad r > 1, \quad t > 0; \quad u(r, 0) = 0; \quad u(1, t) = 1. \quad (7.73)$$

Taking a Laplace transform on time, we find

$$- \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\tilde{u}}{dr} \right) + s\tilde{u} = 0, \quad r > 1; \quad \tilde{u}(1, s) = \frac{1}{s}. $$

Our solution $\tilde{u}$, being a Laplace transform, must be analytic in a right half-plane. Let $\sqrt{s}$ be the square root with positive real part; then $\sqrt{s}$ has a branch on the negative real axis and $\sqrt{s}$ is positive if $s$ is real positive. We can then
write the general solution of the differential equation,

\[ \tilde{u} = A \frac{e^{-r\sqrt{s}}}{r} + B e^{r\sqrt{s}}. \]

Only the first term is bounded, as \( r \to \infty \) in the half-plane \( \text{Re } s > 0 \). Therefore, imposing the boundary condition at \( r = 1 \), we find

\[ \tilde{u} = \frac{e^{-(r-1)\sqrt{s}}}{rs}. \]

By standard techniques of contour integration, we can invert the Laplace transform to obtain

\[ u(r, t) = \frac{1}{r} - \frac{1}{\pi r} \int_{0}^{\infty} \frac{\sin[(r - 1)\sqrt{x}]e^{-xt}}{x} dx, \]

the only contributions being the ones which arise from the branch cut on the negative real axis of the \( s \) plane. It is a simple exercise in differentiation under the integral sign to reduce the last integral to

\[ u(r, t) = \frac{1}{r} - \frac{1}{r} \text{erf} \left[ \frac{(r - 1)/2\sqrt{t}}{r} \right]. \]

Clearly as \( t \to \infty \), \( u(r, t) \to 1/r \) and the steady-state temperature is \( 1/r \), which is a solution of the time-independent equation

\[ \nabla^2 u = 0, \quad r > 1; \quad u \big|_{r=1} = 1. \quad (7.74) \]

A complete asymptotic expansion for large \( t \) can be obtained by using equation (B.10), Appendix B, just as was done following (7.30). Now suppose we had been asked to find the steady temperature in a medium with a spherical hole when the boundary is kept at temperature 1. We would then have been faced with solving (7.74); clearly \( u = 1/r \) is a solution, but so is \( u = 1 \). We cannot choose between these on the basis of steady state alone; the appropriate solution is affected by the nature of the initial condition. If, for instance in (7.73), we had \( u(r, 0) = 1 \) instead of \( u(r, 0) = 0 \), then clearly \( u = 1 \) is the steady (and, in fact, also the transient) solution. Thus we must supply (7.74) with an additional boundary condition at \( r = \infty \), and this condition can be written down only after careful investigation of the physical problem one is really trying to solve.

To illustrate the striking difference between two- and three-dimensional problems consider next the case of an infinite cylindrical hole in an infinite medium. Again we take the initial temperature to be 0 and the temperature on the boundary to be 1. The temperature in the medium will be independent of
the coordinate along the axis of the cylinder and of the polar angle. Thus we have to solve

$$\frac{\partial u}{\partial t} - \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial u}{\partial \rho} \right) = 0, \quad \rho > 1, \quad u(\rho, 0) = 0, \quad u(1, t) = 1, \quad (7.75)$$

where \( \rho \) is the usual radial polar coordinate. The formulation (7.75) should be compared with (7.73). The solution of (7.75) is much more complicated. Taking a Laplace transform of the time coordinate, we find

$$- \frac{1}{\rho} \frac{d}{d\rho} \left[ \rho \frac{d\tilde{u}(\rho, s)}{d\rho} \right] + s\tilde{u}(\rho, s) = 0, \quad \rho > 1, \quad \tilde{u}(1, s) = \frac{1}{s}.$$ 

This differential equation is the modified Bessel equation of order zero (see Appendix B.3, Volume I) and has independent solutions \( I_0(s^{1/2}\rho) \), \( K_0(s^{1/2}\rho) \). Since only \( K_0 \) is bounded at \( \rho = \infty \), we have

$$\tilde{u}(\rho, s) = \frac{1}{s} \frac{K_0(s^{1/2}\rho)}{K_0(s^{1/2})},$$

where \( s^{1/2} \) is the square root of \( s \) which has positive real part. The functions \( K_0(s^{1/2}\rho) \) and \( K_0(s^{1/2}) \) are analytic except for a branch on the negative real \( s \) axis. We can deform the contour of integration in the inversion integral to a loop enclosing the negative real \( s \) axis. Taking into account the branch and the singularity at \( s = 0 \), one obtains

$$u(\rho, t) = 1 - \frac{2}{\pi} \int_0^\infty e^{-\nu^2t} \frac{J_0(\nu\rho)N_0(\nu) - J_0(\nu\rho)N_0(\nu)}{J_0^2(\nu) + N_0^2(\nu)} \frac{dv}{v}. \quad (7.76)$$

The integral term on the right approaches 0 as \( t \to \infty \), so that

$$\lim_{t \to \infty} u(\rho, t) = 1,$$

which should be contrasted with the case of a spherical hole when the limit is \( 1/r \).

As a final remark, we might point out that it is considerably more difficult to find the asymptotic expansion for large \( t \) of (7.76) than for the solution of (7.73). The nature of the singularity of \( \tilde{u}(\rho, s) \) at \( s = 0 \) does not permit us to use the results of Appendix B, but an asymptotic theory can be developed to handle logarithmic singularities such as the one which occurs in the present case at \( s = 0 \).

### 7.4 UNIQUENESS AND CONTINUOUS DEPENDENCE ON THE DATA

Let \( R \) be a bounded region in \( n \) space with boundary \( \sigma \); consider the problem

$$\frac{\partial u}{\partial t} - \nabla^2 u = 0, \quad x \text{ in } R, \quad 0 < t < T \quad (7.77)$$

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with
\[ u(x, 0) = 0, \quad x \text{ in } R; \quad u(x, t) = 0, \quad 0 < t < T, \quad x \text{ on } \sigma. \] (7.78)

These equations describe heat conduction in \( R \) with no sources, zero initial temperature, and zero boundary temperature up to time \( T \). Our physical intuition tells us that the temperature \( u \) is identically zero in \( R \) up to time \( T \) (our prediction stops at time \( T \), since thereafter we know neither the sources nor the boundary temperature). Naturally we hope that the mathematical formulation (7.77), (7.78) will guarantee that \( u(x, t) \equiv 0 \) up to time \( T \). This statement will in fact be proved provided we properly interpret the nature of the boundary and initial conditions (7.78). Some of the pitfalls were already pointed out in connection with the infinite rod. In what follows, \( D \) is the interior of the cylinder in Figure 7.1 and \( \overline{D} \) is the closed cylinder including the lateral surface and bases.

**Theorem 1.** The only solution of (7.77) and (7.78) which is continuous in \( \overline{D} \) is \( u(x, t) \equiv 0 \).

We shall give two proofs. The first is based on energy integrals, the second on a maximum principle.

**First Proof.** Multiply (7.77) by \( \bar{u}(x, t) \), the complex conjugate of \( u \), and integrate over \( R \). Thus
\[ \int_R \bar{u} \frac{\partial u}{\partial t} \, dx - \int_R \bar{u} \nabla^2 u \, dx = 0, \]
or
\[ \frac{1}{2} \frac{d}{dt} \int_R |u|^2 \, dx + \int_R |\text{grad } u|^2 \, dx - \int_\sigma \bar{u} \frac{\partial u}{\partial n} \, dS = 0. \]

Since \( u \) vanishes on \( \sigma \) for \( t < T \),
\[ \frac{1}{2} \frac{d}{dt} \int_R |u|^2 \, dx + \int_R |\text{grad } u|^2 \, dx = 0, \quad 0 < t < T. \]

It follows that
\[ \frac{1}{2} \frac{d}{dt} \int_R |u|^2 \, dx \leq 0, \quad 0 < t < T. \] (7.79)

Now, \( \int_R |u|^2 \, dx \) vanishes at \( t = 0 \) by the initial condition. Therefore, (7.79) shows that, for any \( t > 0 \), \( \int_R |u|^2 \, dx \leq 0 \). Since the integrand is nonnegative and continuous, this implies
\[ u(x, t) = 0, \quad 0 < t < T, \quad x \text{ in } R, \]
or, by continuity,
\[ u(x, t) = 0, \quad (x, t) \text{ in } \overline{D}. \]
REMARK. The proof just given suffers from a minor defect. In using
Green’s theorem to transform \( \int_R \bar{u} \nabla^2 u \, dx \), we need to assume something
about the existence of \( \partial u / \partial n \). On the other hand, we did not use the fact
that the initial values are assumed in the sense of continuity; all that was
required in the proof is
\[
\lim_{t \to 0^+} \int_R |u(x, t)|^2 \, dx = 0;
\]
that is,
\[
\lim_{t \to 0^+} u(x, t) = 0,
\]
in the sense of convergence in the mean.

The next theorem is known as the maximum principle for the heat operator.

Theorem 2. (Maximum Principle). Let \( u(x, t) \) be continuous in the closed
region \( \bar{D} \), and let \( u \) satisfy (7.77). Suppose further that \( u \leq M \) for \( x \) in \( R \), \( t = 0 \),
and for \( x \) on \( \sigma \), \( 0 < t < T \). Then \( u \leq M \) in \( \bar{D} \).

The physical interpretation (see Figure 7.1) is simple. If there are no
sources, the temperature in the interior of \( R \) cannot exceed the maximum of
the initial and boundary temperatures. To prove the maximum principle we
need the preliminary result:

Lemma. Let \( v(x, t) \) be continuous in the region \( \bar{D} \) and let \( v \) satisfy
\[
\frac{\partial v}{\partial t} - \nabla^2 v < 0, \quad (x, t) \text{ in } D.
\] (7.80)
Then the maximum of \( v \) occurs at \( t = 0 \) or on \( \sigma \).

Proof of Lemma. Suppose \( v \) had a maximum at the interior point \( x \) in \( R \),
\( 0 < t < T \); this would imply \( \nabla^2 v \leq 0 \) and \( \partial v / \partial t = 0 \) at the point in question,
which contradicts (7.80). If the maximum occurred at \( x \) in \( R \), \( t = T \), we would
have \( \nabla^2 v \leq 0 \) and \( \partial v / \partial t \geq 0 \), which still violates (7.80).

Proof of Theorem 2. Since \( R \) is bounded it can be enclosed in an \( n \)
sphere of finite radius \( a \). For each \( t > 0 \), the function
\[
v = u + \varepsilon |x|^2
\]
satisfies
\[
\frac{\partial v}{\partial t} - \nabla^2 v = -2n\varepsilon < 0.
\]
From the definition of \( v \),
\[
u \leq v \leq u + \varepsilon a^2,
\]
and since \( v \) satisfies the lemma,

\[
u \leq v \leq M + \varepsilon a^2 \quad \text{for each } \varepsilon > 0.
\]

By letting \( \varepsilon \to 0 \), we find \( u \leq M \), which proves the maximum principle.

**Remark.** Our proof does not exclude the possibility that at an interior point the temperature might equal the maximum of the boundary and initial temperatures. In fact, if the initial temperature and boundary temperature are equal to the same constant, then \( u \) will be this constant for all \( t \leq T \).

**Corollary (Minimum Principle).** Let \( u(x, t) \) be continuous in the closed region \( \overline{D} \) and let \( u \) satisfy (7.77). Suppose that \( u \geq m \) for \( x \in R, t = 0 \), and for \( x \in \sigma, 0 < t < T \). Then \( u \geq m \) for all \((x, t)\) in \( \overline{D} \).

**Proof.** Apply the maximum principle to the function \(-u\).

From the maximum and minimum principles, we conclude that if there are no sources in \( R \), the temperature in the interior must lie between the maximum and minimum temperatures on the boundary and at \( t = 0 \).

**Corollary 1.** Let \( u_1, u, u_2 \) be solutions of the homogeneous heat equation for \( \{x \in R; 0 < t < T\} \) where \( R \) is a bounded region. If the initial and boundary values satisfy the inequality \( u_1 \leq u \leq u_2 \), then, for all \( x \in \overline{R} \) and all \( t, 0 \leq t \leq T \), we have

\[
u_1(x, t) \leq u(x, t) \leq u_2(x, t).
\]

**Proof.** Use the maximum principle for \( u - u_1 \) and \( u_2 - u \).

**Theorem 3 (Uniqueness Theorem).** Let \( q(x, t) \) and \( v(x, t) \) be continuous in \( \overline{D} \) and let \( h(x, t) \) be continuous for \( \{x \in \sigma, 0 \leq t \leq T\} \); let \( f(x) \) be continuous for \( x \in \overline{R} \). Moreover, let the initial data be compatible with the boundary data in the sense of continuity, that is, for \( x \) on \( \sigma \), \( f(x) = h(x, 0) \). Then the problem

\[
\frac{\partial v}{\partial t} - \nabla^2 v = q(x, t), \quad (x, t) \in D; \tag{7.81}
\]

\[
v(x, 0) = f(x), \quad x \in R; \quad v(x, t) = h(x, t), \quad x \in \sigma, \quad 0 < t < T;
\]

has at most one solution.

**Proof.** If there were two solutions \( v_1 \) and \( v_2 \), their difference \( u(x, t) \) would be continuous in \( \overline{D} \) and would vanish at \( t = 0 \) and also on the boundary \( \sigma \) for \( 0 \leq t \leq T \). By the maximum and minimum principles, this implies that \( u \) is identically 0 in \( \overline{D} \).
Theorem 4 (Continuous Dependence on Initial and Boundary Data). Let \( v_1 \) and \( v_2 \) be the solutions of (7.81) for the same \( q \) but with data \( f_1, h_1, \) and \( f_2, h_2, \) respectively. Suppose
\[
\max_{x \in R} |f_1 - f_2| < \varepsilon, \quad \max_{x \text{ on } \sigma} |h_1 - h_2| < \varepsilon.
\]
Then
\[
\max_{x \in R} |v_1 - v_2| < \varepsilon.
\]

Proof. Direct application of maximum and minimum principle.

The proof of existence of a solution of (7.81) is much more difficult and beyond the scope of the present book. Usually we find an explicit solution by one of the methods of the preceding sections and then check that (7.81) is satisfied. The form of the explicit solution also frequently enables us to prove continuous dependence on the source term \( q \). To prove uniqueness for unbounded regions we must make some further requirement as to the behavior of the solution for large \( |x| \). We shall limit ourselves to the case where the solution is bounded, although more delicate arguments can be used to extend the proof to solutions which have exponential behavior at infinity.

Theorem 5. Let \( R \) be an unbounded region and let \( u(x, t) \) be continuous and bounded in \( \{ x \in \overline{R}; \, 0 \leq t \leq T \} \). Then the only solution of
\[
\frac{\partial u}{\partial t} - \nabla^2 u = 0, \quad x \text{ in } R, \quad 0 < t < T;
\]
\[
 u(x, 0) = 0, \quad x \text{ in } R; \quad u(x, t) = 0, \quad x \text{ on } \sigma, \quad 0 < t < T,
\]
is \( u(x, t) \equiv 0 \).

Proof. Let \( |u(x, t)| \leq M \) and consider the space region \( R_0 \) which is the intersection of \( R \) and a large sphere of radius \( r_0 \). The functions
\[
v_1(x, t) = -\frac{2M}{r_0^2} \left( \frac{|x|^2}{2n} + t \right), \quad v_2(x, t) = \frac{2M}{r_0^2} \left( \frac{|x|^2}{2n} + t \right)
\]
are both solutions of the homogeneous heat equation. In \( R_0 \) we have
\[
v_1(x, 0) \leq u(x, 0) = 0 \leq v_2(x, 0),
\]
and, when \( x \) is on the boundary of \( R_0, \, 0 \leq t \leq T, \)
\[
v_1(x, t) \leq u(x, t) \leq v_2(x, t).
\]
By Corollary 1 it follows that
\[
v_1(x, t) \leq u(x, t) \leq v_2(x, t), \quad x \text{ in } \overline{R_0}, \quad 0 \leq t \leq T.
\]
For a fixed \( (x, t) \) we let \( r_0 \to \infty \); from the explicit expressions for \( v_1 \) and \( v_2 \), it follows that \( u(x, t) = 0 \).
§7.5 MISCELLANEOUS TOPICS RELATED TO THE HEAT EQUATION

Corollary. There is at most one bounded solution of (7.81) even if $R$ is an unbounded region.

Next we return to the case of a bounded region $R$ and discuss the behavior of (7.81) as $t \to \infty$. If the source term $q(x, t)$ and the boundary term $h(x, t)$ approach the limits $q_\infty(x)$ and $h_\infty(x)$, we expect that, independent of the initial temperature, $u(x, t)$ should approach the steady-state temperature $u_\infty(x)$ unambiguously defined from

$$-\nabla^2 u_\infty = q_\infty(x), \quad x \text{ in } R; \quad u_\infty(x) = h_\infty(x), \quad x \text{ on } \sigma. \quad (7.82)$$

Of course we have already seen that this result cannot hold for an unbounded region without some qualifications on the behavior at infinity of the initial values.

We state without proof the following theorem, which makes precise the approach to steady state.

Theorem 6. Let $R$ be a bounded region and let $u(x, t)$ be the solution of (7.81). If $q(x, t) \to q_\infty(x)$ uniformly in $\bar{R}$ and $h(x, t) \to h_\infty(x)$ uniformly on $\sigma$, then $u(x, t)$ approaches $u_\infty(x)$ uniformly in $\bar{R}$.

7.5 MISCELLANEOUS TOPICS RELATED TO THE HEAT EQUATION

Semigroups of Operators

Consider heat conduction in an infinite rod without sources. The initial temperature $u(x, 0)$ determines the temperature $u(x, t)$ for all $t > 0$ by the formula

$$u(x, t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-(x-y)^2/4t} u(y, 0) dy. \quad (7.83)$$

If the initial temperature is bounded, so is $u(x, t)$. We can write, with $t$ fixed,

$$u(x, t) = S_t u(x, 0),$$

where $S_t$ is a linear transformation whose domain is the set $B$ of bounded functions of $x$, $-\infty < x < \infty$, and whose range is a subset of $B$. The transformation $S_t$ depends parametrically on $t$. By definition, $S_0$ transforms $u(x, 0)$ into itself and is therefore the identity transformation. We observe that

$$\lim_{t \to 0^+} S_t = S_0 = I,$$

so that $S_t$ depends continuously on the parameter $t$. The temperature at the time $t + \tau$ is given by

$$u(x, t + \tau) = S_{t+\tau} u(x, 0). \quad (7.84)$$

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Alternatively, we can first find the temperature at the intermediate time $t$ from

$$u(x, t) = S_t u(x, 0),$$

and use $u(x, t)$ as an initial value to calculate the temperature $\tau$ units of time later. Thus

$$u(x, t + \tau) = S_\tau u(x, t) = S_\tau S_t u(x, 0).$$

Similarly, we have

$$u(x, t + \tau) = S_\tau S_t u(x, 0).$$

Therefore, comparing with (7.84), we obtain

$$S_{t + \tau} = S_t S_\tau = S_\tau S_t, \quad t \geq 0, \quad \tau \geq 0. \quad (7.85)$$

We say that the operators $S_t$ form a semigroup. The identity element in the semigroup is $S_0$. The reason that we do not have a group is that the inverse element does not exist for all functions in the set $B$.

It should be noted that the relation (7.85) can be obtained only because the physical parameters are time-invariant.

From (7.85) it follows that

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-(x-y)^2/4t} \frac{1}{\sqrt{4\pi \tau}} e^{-(y-\xi)^2/4\tau} \, dy$$

$$= \frac{1}{\sqrt{4\pi (t + \tau)}} e^{-(x-\xi)^2/4(t + \tau)}, \quad t, \tau \geq 0, \quad (7.86)$$

a relation which can be verified independently by performing the required integration via completion of the square of the exponent. In terms of the Green's function notation, (7.86) states that

$$g(x, t + \tau | \xi, 0) = \int_{-\infty}^{\infty} g(x, t + \tau | y, t) g(y, t | \xi, 0) \, dy$$

$$= \int_{-\infty}^{\infty} g(x, \tau | y, 0) g(y, t | \xi, 0) \, dy, \quad (7.87)$$

a formula which is valid for the causal Green's function for any region $R$, if we replace the integration from $-\infty$ to $\infty$ by an integration over $R$.

**Ill-Posed Problems in Heat Conduction—The Backward Heat Equation**

The function $u(x, t)$ of (7.83) describes the evolution of the temperature in the rod as the parameter $t$ increases. We may think of $u(x, t)$ as a continuous series of snapshots of the temperature of the rod. If we now run the film
backward, the new series of snapshots (in reverse order from the original one) is a function $v(x, t)$ which satisfies the backward heat equation

$$-\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = 0.$$

Thus the initial value problem for the backward heat equation is entirely equivalent to a terminal value problem for the original or forward heat equation. We now consider this latter problem; that is, given $u(x, t)$ for some specific $t > 0$, can we determine the initial temperature $u(x, 0)$ which would have given rise to $u(x, t)$ through the process of heat conduction? It is no restriction to set $t = 1$; letting $u(x, 0) = a(x)$ and $u(x, 1) = b(x)$, our problem is to find $a(x)$ given $b(x)$. Now, from (7.83), we have

$$b(x) = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} e^{-(x-\xi)^2/4} a(\xi) d\xi, \quad -\infty < x < \infty. \tag{7.88}$$

To find $a(x)$, we must solve the integral equation (7.88), with $b(x)$ given. Since the right side smooths out any initial function $a(x)$, no solution is possible unless $b(x)$ is infinitely differentiable! If a solution exists it can be shown to be unique; that is, there is at most one initial temperature which yields a given temperature 1 unit of time later. The solution $a(x)$ does not depend continuously on the data $b(x)$. To see this we observe that the solution of the forward heat equation with initial temperature $\varepsilon \sin \alpha x$ (where $\alpha$ and $\varepsilon$ are real constants) is $\varepsilon e^{-\alpha^2 t} \sin \alpha x$, so that the temperature at $t = 1$ is $\varepsilon e^{-\alpha^2} \sin \alpha x$. Therefore the one and only solution of (7.88) with $b(x) = \varepsilon \sin \alpha x$ is

$$a(x) = \varepsilon e^{\alpha^2} \sin \alpha x.$$

Now, for any $\alpha$,

$$\max_{-\infty < x < \infty} |b(x)| = \varepsilon,$$

but

$$\max_{-\infty < x < \infty} |a(x)| = \varepsilon e^{\alpha^2},$$

which can be made as large as we wish by choosing $\alpha$ large enough. Therefore a "small" $b(x)$ can give us an arbitrarily "large" $a(x)$, and the terminal value problem for the heat equation (or initial value problem for the backward heat equation) is ill posed.

Similar properties are exhibited by the problem of the finite rod. If the ends are at zero temperature and there are no sources, then the temperature $b(x)$ at $t = 1$ can be expressed by (7.56),

$$b(x) = \sum_{n=1}^{\infty} \frac{2}{l} e^{-n^2 \pi^2 t/l^2} \left[ \int_{0}^{l} a(x) \sin \frac{n \pi x}{l} \, dx \right] \sin \frac{n \pi x}{l}, \tag{7.89}$$

or

$$b(x) = \int_{0}^{l} g(x, 1 | \xi, 0) a(\xi) d\xi, \tag{7.90}$$
where
\[ g(x, 1 | \xi, 0) = \frac{2}{l} \sum_{n=1}^{\infty} e^{-2n^2 \pi^2 / l^2} \sin \frac{n\pi x}{l} \sin \frac{n\pi \xi}{l}. \]

We may regard (7.90) as an integral equation for \( a(x) \) when \( b(x) \) is given. Let the Fourier sine coefficients of the given function \( b(x) \) be \( \{b_n\} \) and those of the unknown function \( a(x) \) be \( \{a_n\} \). Then from (7.89),
\[ b_n = e^{-n^2 \pi^2 / l^2} a_n, \]
or
\[ a_n = b_n e^{n^2 \pi^2 / l^2}. \]

Now if \( b(x) \) is arbitrary, all we can say about \( b_n \) is that \( \sum |b_n|^2 < \infty \). The presence of the exponentially increasing factor in \( a_n \) makes it highly unlikely that \( \sum |a_n|^2 \) converges, so that (7.90) does not have an \( L_2 \) solution for \( a(x) \). The necessary and sufficient condition for (7.90) to have a solution is that
\[ \sum_{n=1}^{\infty} e^{2n^2 \pi^2 / l^2} |b_n|^2 < \infty, \quad (7.91) \]
which in turn implies that \( b(x) \) must be infinitely differentiable. If the condition (7.91) is satisfied, then (7.90) has the one and only solution
\[ a(x) = \sum_{n=1}^{\infty} b_n e^{n^2 \pi^2 / l^2} \sin \frac{n\pi x}{l}. \]

The problem of finding \( a(x) \) from \( b(x) \) is not well posed. In fact, if \( b(x) = \varepsilon \sin (m \pi x / l) \), then \( \max_{0 < x < l} |\mu(x, 1)| = \varepsilon \), but
\[ a(x) = \varepsilon e^{m^2 \pi^2 / l^2} \sin \frac{m\pi x}{l}, \]
\[ \max_{0 < x < l} |a(x)| = \varepsilon e^{m^2 \pi^2 / l^2}, \]
which is arbitrarily large for \( m \) large enough. Thus even if \( b(x) \) is "small," \( a(x) \) may be very large; in other words, the solution does not depend continuously on the data.

**Diffusion Interpretation and the Fokker-Planck Equation**

Suppose a medium \( R \) is filled with a solvent and that there is an initial concentration \( \mu(x, 0) \) of a solute. Then the concentration \( \mu(x, t) \) satisfies the diffusion equation
\[ \frac{\partial \mu}{\partial t} - \nabla^2 \mu = 0. \]

The same equation also occurs in the Brownian motion of microscopic particles. Then \( \mu(x, t) \) represents the probability density per unit volume at
time $t$; that is, $u(x, t)dx$ is the probability of finding the particle at time $t$ in a volume element $dx$ at $x$.

In either interpretation the boundary condition $\partial u/\partial n = 0$ on $\sigma$ characterize a reflecting boundary, whereas $u = 0$ represents an absorbing boundary. For the first boundary condition, the solute remains in $R$, whereas for the second boundary condition, the solute is removed from $R$ as soon as it reaches the boundary.

If a steady drift is superposed on the motion, say in the $l$ direction, then the equation is altered:

$$\frac{\partial u}{\partial t} + 2\gamma l \cdot \text{grad} u - \nabla^2 u = 0,$$  \hspace{1cm} (7.92)

where $2\gamma$ is the drift velocity. This equation is a simple form of the Fokker-Planck equation and can easily be transformed to the usual diffusion equation by a change of variables. Consider, for instance, the one-dimensional case

$$\frac{\partial u}{\partial t} + 2\gamma \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = 0,$$  \hspace{1cm} (7.93)

and let

$$u = ve^{\gamma x}e^{-\gamma t};$$  \hspace{1cm} (7.94)

then (7.93) becomes

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = 0.$$

As a particular problem in connection with (7.93), we ask for the concentration in an infinite medium, when initially the solute is localized at $x = x_0$. Thus we want to solve (7.93) for the initial condition $u(x, 0) = \delta(x - x_0)$. Making the transformation (7.94), we find

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = 0, \quad -\infty < x < \infty, \quad t > 0,$$

$$v(x, 0+) = e^{-\gamma x_0}\delta(x - x_0),$$

and, therefore, by our results for the heat equation,

$$v(x, t) = e^{-\gamma x_0} \frac{e^{-(x-x_0)^2/4t}}{(4\pi t)^{1/2}}.$$

Consequently,

$$u(x, t) = e^{\gamma(x-x_0)}e^{-\gamma^2t} \frac{e^{-(x-x_0)^2/4t}}{(4\pi t)^{1/2}}.$$  \hspace{1cm} (7.95)

**Asymptotic Formula for the Eigenvalues of the Negative Laplacian**

In (7.57) we showed that the Green's function for the heat equation in a region $R$ can be expanded in the complete set of orthonormal eigenfunctions
\{\varphi_i\} defined from
\[-\nabla^2 \varphi_i = \lambda_i \varphi_i, \quad x \text{ in } R; \quad \varphi_i(x) = 0, \quad x \text{ on } \sigma. \quad (7.96)\]

The Green's function for the heat operator corresponding to a unit source
introduced at \(t = 0, \ x = x_0\), is given by
\[g(x, \ t \mid x_0, \ 0) = \sum_{k=1}^{\infty} \varphi_k(x)\varphi_k(x_0)e^{-\lambda_k t}. \quad (7.97)\]

We now want to change our point of view so as to try to obtain information
about the \{\lambda_k\} from (7.97). If we let \(x = x_0\) and integrate over \(R\), we find
\[\int_R g(x, t \mid x, 0)dx = \sum_{k=1}^{\infty} e^{-\lambda_k t}. \quad (7.98)\]

Estimation of the left side for small \(t\) will give us asymptotic formulas for
the distribution of the eigenvalues of (7.96). In what follows we restrict ourselves
to a two-dimensional region \(R\). We claim that for small \(t, g(x, t \mid x_0, 0)\)
is indistinguishable from the free-space fundamental solution \(C(x, t \mid x_0, 0)\)
given by
\[C(x, t \mid x_0, 0) = \frac{1}{4\pi t} e^{-|x-x_0|^2/4t}. \]

In fact, \(C\) characterizes the concentration at time \(t\) in an infinite medium when
the solute is initially concentrated at \(x = x_0\); on the other hand, \(g\) gives the
concentration for a finite medium with an absorbing boundary under the
same initial condition. It is physically obvious that if \(t\) is small, the absorbing
boundary is not felt and we must have
\[g(x, t \mid x_0, 0) \sim \frac{1}{4\pi t} e^{-|x-x_0|^2/4t}, \quad t \text{ small.}\]

One can easily be more precise. We can write
\[g(x, t \mid x_0, 0) = C(x, t \mid x_0, 0) - \nu(x, t, x_0), \]
where \(\nu\) satisfies the homogeneous heat equation in \(R\) in the variables \((x, t)\). Moreover, at \(t = 0+\),
\[\nu(x, 0+, x_0) = C(x, 0+ \mid x_0, 0) - g(x, 0+ \mid x_0, 0) = \delta(x - x_0) - \delta(x - x_0) = 0. \]

On the boundary \(\sigma\) of \(R,\)
\[\nu(x, t, x_0) = C(x, t \mid x_0, 0) > 0. \]

Therefore by the maximum principle, we know that \(\nu\) is positive for \(t > 0\)
and we can actually say that
\[g(x, t \mid x_0, 0) < C(x, t \mid x_0, 0). \]
Moreover, 
\[
\lim_{t \to 0^+} g(x, t | x_0, 0) = C(x, 0+ | x_0, 0).
\]
In view of this result, we can replace \( g \) by \( C \) in (7.98) for small \( t \). Thus
\[
\sum_{k=1}^{\infty} e^{-\lambda_k t} \sim \int_{\mathcal{R}} \frac{1}{4\pi t} \; dx = \frac{A_R}{4\pi t},
\]  
(7.99)
where \( A_R \) is the area of the plane region \( \mathcal{R} \).

We want to rewrite (7.99) so as to take advantage of certain asymptotic properties of Laplace transforms. With \( H \) the Heaviside function, we note that
\[
t \int_0^{\infty} H(\lambda - \lambda_k) e^{-t\lambda} \; d\lambda = t \int_{\lambda_k}^{\infty} e^{-t\lambda} \; d\lambda = e^{-t\lambda_k},
\]
and, therefore,
\[
t \int_0^{\infty} \sum_{k=1}^{\infty} H(\lambda - \lambda_k) e^{-t\lambda} \; d\lambda = \sum_{k=1}^{\infty} e^{-t\lambda_k}.
\]
Now let
\[
N(\lambda) = \sum_{k=1}^{\infty} H(\lambda - \lambda_k),
\]
where \( N(\lambda) \) is the number of eigenvalues less than \( \lambda \). Then (7.99) becomes
\[
\int_0^{\infty} N(\lambda) e^{-t\lambda} \; d\lambda \sim \frac{A_R}{4\pi t^2}, \quad t \text{ small},
\]  
(7.100)
where \( t \) plays the role of the transform variable. By Appendix B, we infer from (7.100) the asymptotic behavior of \( N(\lambda) \) for large values of \( \lambda \):
\[
N(\lambda) \sim \frac{\lambda A_R}{4\pi}.
\]  
(7.101)

The formula (7.101) has a glorious history, in which the name Weyl has the most important place. His proof was based on the theory of integral equations, but simpler proofs were given later by using variational methods. The physical content of (7.101) is remarkable; it essentially states that the asymptotic distribution of the frequencies of a membrane depends only on its area and not on its shape.

The expression (7.101) is only the leading term of an asymptotic expansion. The next term can be obtained by substituting a better approximation for \( g \) in (7.98) consisting of the free-space causal solution plus a correction term; when \( t \) is small, this correction is due principally to the absorption at the closest point on the boundary. Thus we can write
\[
g(x, t | x_0, 0) \sim \frac{1}{4\pi t} e^{-|x-x_0|^2/4t} - \frac{1}{4\pi t} e^{-|x-x_0|^2/4t},
\]
where \( x_0^* \) is the image of \( x_0 \) with respect to a tangent line to the boundary at the point on \( \sigma \) closest to \( x_0 \). Therefore,

\[
g(x, t | x, 0) \sim \frac{1}{4 \pi t} - \frac{1}{4 \pi t} e^{-h^2(x)/t},
\]

where \( h(x) \) is the distance from \( x \) to the nearest point on the boundary. From (7.98), we find, for small \( t \),

\[
\sum_{k=1}^{\infty} e^{-\lambda_k t} \sim \frac{A_R}{4 \pi t} - \frac{1}{4 \pi t} \int_R e^{-h^2(x)/t} \, dx.
\]

For \( t \) small, the principal contribution to the integral stems from those points \( x \), where \( h^2(x) \) is small, that is, from a small strip at the boundary \( \sigma \) of \( R \). Considering a strip of constant small width \( h_0 \), we have

\[
\int_R e^{-h^2(x)/t} \, dx = \int_0^{h_0} e^{-h^2/t} L_R \, dh,
\]

where \( L_R \) is the perimeter of \( R \). Making the change of variables \( z = h/t^{1/2} \), we find

\[
\int_R e^{-h^2(x)/t} \, dx = \sqrt{t} L_R \int_0^{h_0/t^{1/2}} e^{-z^2} \, dz \sim \sqrt{t} L_R \int_0^{\infty} e^{-z^2} \, dz = \sqrt{t} \pi L_R^2 / 2.
\]

Therefore,

\[
\sum_{k=1}^{\infty} e^{-\lambda_k t} \sim \frac{A_R}{4 \pi t} - \frac{L_R}{8 \sqrt{\pi} t},
\]

\[
\int_0^{\infty} N(\lambda) e^{-\lambda t} \, d\lambda \sim \frac{A_R}{4 \pi t^2} - \frac{L_R}{8 \pi^{1/2} t^{3/2}},
\]

so that

\[
N(\lambda) \sim \frac{\lambda A_R}{4 \pi} - \frac{L_R}{4 \pi} \sqrt{\lambda}, \tag{7.102}
\]

which is a more delicate asymptotic formula than (7.101).

**Composite Medium**

We shall consider a typical problem for a composite medium. In three dimensions, let the region \( r < 1 \) be filled with a homogeneous medium with conductivity \( k_1 \) and diffusivity \( a_1 \), and let the region \( r > 1 \) be filled with another homogeneous medium of conductivity \( k_2 \) and diffusivity \( a_2 \). We assume that the media are in intimate contact so that the boundary conditions (6.163) and (6.164) hold. Let us find the temperature in space-time when the initial temperature is 1 for \( r < 1 \), 0 for \( r > 1 \). The temperature \( u \) will then depend only
on r and t; let \( u_1(r, t) \) be the temperature in \( r < 1 \) and \( u_2(r, t) \) in \( r > 1 \). Then we have to solve the boundary value problem

\[
\frac{\partial u_1}{\partial t} - \frac{a_1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial u_1}{\partial r} \right) = 0, \quad r < 1, \quad t > 0 \quad u_1(r, 0) = 1, \quad r < 1;
\]

\[
\frac{\partial u_2}{\partial t} - \frac{a_2}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial u_2}{\partial r} \right) = 0, \quad r > 1, \quad t > 0; \quad u_2(r, 0) = 0, \quad r > 1;
\]

with the interface conditions

\[ u_1(1, t) = u_2(1, t), \quad t > 0 \]

\[ k_1 \frac{\partial u_1}{\partial r} (1, t) = k_2 \frac{\partial u_2}{\partial r} (1, t), \quad t > 0. \]

Let \( \tilde{u}_1(r, s) \) and \( \tilde{u}_2(r, s) \) be the Laplace transforms \( u_1(r, t) \) and \( u_2(r, t) \), respectively. Taking the transform of the differential equations, we find

\[
s\tilde{u}_1 - \frac{a_1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\tilde{u}_1}{dr} \right) = 1,
\]

\[
s\tilde{u}_2 - \frac{a_2}{r^2} \frac{d}{dr} \left( r^2 \frac{d\tilde{u}_2}{dr} \right) = 0.
\]

These equations are easily solved by making a substitution of the form \( \tilde{u} = v/r \). Let \( s^{1/2} \) be the square root which has positive real part unless \( s \) is real negative; taking into account the fact that \( \tilde{u}_1 \) is bounded at \( r = 0 \) and \( \tilde{u}_2 \) at \( r = \infty \), we obtain

\[
\tilde{u}_1 = \frac{1}{s} + B \frac{\sinh(\sqrt{s/a_1}r)}{r};
\]

\[
\tilde{u}_2 = \frac{A}{r} \exp(-\sqrt{s/a_2}r).
\]

The interface conditions give

\[
B = \frac{1}{s} \left[ \frac{k_1}{k_2} \left( \sinh \sqrt{s/a_1} - \sqrt{s/a_1} \cosh \sqrt{s/a_1} \right) \right. \\
\left. \times (1 + \sqrt{s/a_2})^{-1} - \sinh \sqrt{s/a_1} \right]^{-1};
\]

\[
A = \exp \left( \sqrt{s/a_2} \left[ \frac{1}{s} + B \sinh \sqrt{s/a_1} \right] \right).
\]
We shall be interested only in the temperature at $r = 0$. We must then invert $\tilde{u}_1(0, s)$, where

$$
\tilde{u}_1(0, s) = \frac{1}{s} + B \sqrt{s/a_1}
$$

$$
= \frac{1}{s} + \frac{1}{s} (1 + \sqrt{s/a_2}) \left[ \frac{k_1}{k_2} \left( \frac{\sinh \sqrt{s/a_1}}{\sqrt{s/a_1}} - \cosh \sqrt{s/a_1} \right) \right]
$$

$$
- \frac{\sinh \sqrt{s/a_1}}{\sqrt{s/a_1}} (1 + \sqrt{s/a_2})^{-1}
$$

$$
= \frac{1}{s} - \frac{1}{s} (1 + \sqrt{s/a_2}) \left[ 1 - \sum_{n=1}^{\infty} \alpha_n \left( \frac{s}{a_1} \right)^n + \sum_{n=0}^{\infty} \beta_n \frac{s^{n+1/2}}{a_1^n a_2^{1/2}} \right]^{-1},
$$

where

$$
\alpha_1 = -\frac{k_1}{3k_2} - \frac{1}{6},
$$

$$
\beta_0 = 1, \quad \beta_1 = 1/6.
$$

Let

$$
f(s) = \sum_{n=1}^{\infty} \alpha_n \left( \frac{s}{a_1} \right)^n - \sum_{n=0}^{\infty} \beta_n \frac{s^{n+1/2}}{a_1^n a_2^{1/2}};
$$

then $f(0) = 0$, and, for small $s$, we may expand $[1 - f(s)]^{-1}$ in a Maclaurin series. This yields

$$
\tilde{u}(0, s) = \frac{1}{s} - \frac{1}{s} \left[ 1 + f(s) + f^2(s) + f^3(s) + \cdots \right]
$$

$$
- \frac{1}{a_1^{1/2} s^{1/2}} \left[ 1 + f(s) + f^2(s) + f^3(s) + \cdots \right].
$$

We note that the term in $1/s$ has 0 coefficient. We need not be concerned with other integral powers of $s$ in calculating the asymptotic value for large $t$ [see Appendix B, equation (B.10)]. The leading terms are the ones in $s^{-1/2}$ and $s^{1/2}$. A slightly tedious but straightforward calculation shows that the coefficient of $s^{-1/2}$ vanishes and that the leading term is

$$
\tilde{u}(0, s) \sim -\frac{k_1}{3k_2} \frac{s^{1/2}}{a_1 a_2^{1/2}}.
$$

On inversion, we have

$$
u(0, t) \sim \frac{1}{6\pi^{1/2}} \frac{k_1}{k_2} \frac{t^{-3/2}}{a_1 a_2^{1/2}}.
$$

(7.103)
If the media are the same, we find, with the common diffusivity denoted by \( a \),
\[
u(0, t) \sim \frac{1}{6\pi^{1/2}a^{3/2}t^{3/2}}. \tag{7.104}
\]

For such a homogeneous single medium, the temperature could be calculated directly for large \( t \) by the following argument. The initial temperature is equivalent to a point source at \( r = 0 \) of strength \( 4\pi \), and, by (5.140), the temperature for \( t > 0 \) is
\[
\frac{4\pi}{(4\pi at)^{3/2}},
\]
which coincides with (7.104).

**The Stefan Problem**

The following physical situation is representative of a simple class of heat-conduction problems involving a change of phase. Let the half-space \( x > 0 \) be filled with ice at temperature 0 degrees centigrade. At \( t = 0 \) the wall \( x = 0 \) is placed and kept thereafter at the constant positive temperature \( U \). It is clear that the ice near the wall will start melting and that the melting front will propagate as \( t \) increases. Figure 7.6 shows the front described by the curve \( x = \xi(t) \) in space-time. Conditions are clearly independent of \( y \) and \( z \). For \( x > \xi(t) \), the phase is ice at zero temperature; for \( x < \xi(t) \) the phase is water with an unknown temperature (between 0 and \( U \)) depending on position and time. The curve \( x = \xi(t) \) is unknown and must be found as part of the solution. In \( x < \xi(t) \) we have heat conduction in water. On the phase interface \( x = \xi(t) \), we must have zero temperature. Moreover, we can set up a heat balance in time \( \Delta t \) for the elementary sheet of ice being melted during that time. The front has moved an amount \( \Delta \xi \) in time \( \Delta t \), which means that the
mass melted per unit area in the $yz$ plane is $\rho \Delta \xi$, where $\rho$ is the mass density. If $v$ is the latent heat of melting, we must supply by conduction an amount of heat $v\rho \Delta \xi$ per unit area to cause the melting. Since the gradient of the temperature is 0 in the ice phase, we have

$$-k \frac{\partial u}{\partial x} \bigg|_{x=\xi} \Delta t = v\rho \Delta \xi,$$  

(7.105)

where $u(x, t)$ is the water temperature. Putting all our information together, we have to solve the boundary value problem

$$\frac{\partial u}{\partial t} - a \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < x < \xi(t), \quad t > 0;$$

$$u(0, t) = U; \quad u(\xi, t) = 0; \quad -k \frac{\partial u}{\partial x}(\xi, t) = v\rho \xi'(t).$$  

(7.106)

We emphasize once more that $\xi(t)$ is unknown and must be found as part of the solution. Such problems are called free boundary problems and, as we shall see, these are nonlinear problems. Because the temperature at $x = 0$ is kept at the constant value $U$, the solution is atypically simple. We proceed by observing that, for an infinite medium with diffusivity $a$, the function

$$A + B \text{ erf} \frac{x}{2\sqrt{at}}$$

is a solution of the homogeneous heat equation. To satisfy the additional conditions in (7.106) by a solution of this form, we need

$$A = U \quad A + B \text{ erf} \frac{\xi}{2\sqrt{at}} = 0; \quad -Bk \frac{1}{(\pi t)^{1/2}} e^{-\xi^2/4at} = v\rho \xi'(t).$$

The second of these conditions implies that $\xi$ is proportional to $\sqrt{t}$, that is,

$$\xi(t) = 2\alpha \sqrt{at},$$  

(7.107)

where $\alpha$ is an unknown constant. Thus the second and third conditions become

$$U + B \text{ erf} \alpha = 0; \quad -\frac{Bk}{(\pi t)^{1/2}} e^{-\alpha^2} = \alpha v\rho \left(\frac{a}{t}\right)^{1/2}. $$

Since $B = -U/\text{ erf} \alpha$, we determine $\alpha$ from

$$\frac{e^{-\alpha^2}}{\text{ erf} \alpha} = \frac{v\rho a}{kU} \pi^{1/2} \alpha,$$

an equation which has one and only one solution. After solving for $\alpha$, we obtain the equation for the transition front from (7.107) and then

$$u(x, t) = U - \frac{U}{\text{ erf} \alpha} \text{ erf} \left(\frac{x}{2\sqrt{at}}\right), \quad 0 < x < \xi(t).$$
EXERCISES

7.4 Separate the heat equation
\[ \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \]
for the case of a ring \(-\frac{1}{2} < x < \frac{1}{2}\). Show that
\[ u(x, t) = \sum_{n = -\infty}^{\infty} f_n e^{i2\pi nx} e^{-\lambda_n^2 n^2 t}, \]
where
\[ f_n = \int_{-\frac{1}{2}}^{1/2} f(x) e^{-i2\pi nx} \, dx, \]
with \(f(x)\) the initial temperature. If \(f(x) = \delta(x)\), obtain the result (7.48).

7.5 Let the half-space \(x > 0\) be filled with ice at the constant temperature \(V < 0\) (instead of \(V = 0\) as in the example of the Stefan problem in the text). At \(t = 0\) the wall \(x = 0\) is placed and kept thereafter at the constant positive temperature \(U\). If \(k_1\) and \(a_1\) are the conductivity and diffusivity for water and \(k_2\) and \(a_2\) the ones for ice, show that
\[ \frac{\partial u_1}{\partial t} - a_1 \frac{\partial^2 u_1}{\partial x^2} = 0, \quad 0 < x < \xi(t); \quad u_1(0, t) = U, \]
\[ \frac{\partial u_2}{\partial t} - a_2 \frac{\partial^2 u_2}{\partial x^2} = 0, \quad \xi(t) < x < \infty; \quad u_2(x, 0) = V, \]
\[ u_1(\xi, t) = u_2(\xi, t) = 0; \quad -k_1 \frac{\partial u_1}{\partial x} (\xi -, t) + k_2 \frac{\partial u_2}{\partial x} (\xi +, t) = \nu \rho \xi'(t), \]
where \(u_1\) is the water temperature and \(u_2\) the temperature of the ice. Calculate \(\xi(t)\), \(u_1\), and \(u_2\) by a method similar to the one in the text.

7.6 A cylindrical tank of height \(h\) is filled with water at the constant temperature \(U > 0\). The free surface is then kept at the constant temperature \(V < 0\). Assuming that the base and lateral surface of the tank are insulated, and neglecting any expansion due to phase change, determine how long it takes for all the water to turn to ice. How would you treat the problem if the base were kept at a temperature \(W, 0 < W < U\)?

7.7 Derive the following asymptotic formula for the eigenvalues of (7.96) when \(R\) is a bounded three-dimensional region:
\[ N(\lambda) \sim \frac{V_R \lambda^{3/2}}{6\pi^2} - \frac{S_R \lambda}{16\pi}, \]
where \(N(\lambda)\) is the number of eigenvalues less than \(\lambda\), and \(V_R\) and \(S_R\) are the volume and surface area, respectively, of the region \(R\).
By explicitly calculating the eigenvalues for a cube, check the leading term in the asymptotic formula for \( N(\lambda) \).

7.8 Derive asymptotic formulas for the eigenvalues of (7.96) for a bounded two-dimensional region with (a) the boundary condition \( \partial \phi / \partial n = 0 \), and (b) the boundary condition \( (\partial \phi / \partial n) + \theta \phi = 0 \), where \( \theta \) is a positive constant.

7.9 Let \( R \) be a bounded region in \( n \)-space.

(a) Show that if \( \nabla^2 v > 0 \) in \( R \), the maximum of \( R \) occurs on the boundary.

(b) Set \( v = u + \epsilon |x|^2 \) and use part (a) to prove that the maximum value of the solution of \( \nabla^2 u = 0 \) occurs on the boundary and hence that the minimum value also occurs on the boundary. This then is another way of proving the maximum principle for harmonic functions.

7.10 Consider a rod of length \( l \) whose ends are kept at zero temperature. The initial temperature is that due to a unit dipole at the midpoint of the rod and there are no sources for \( t > 0 \). Show that

\[
  u(x, t) = \sum_{n=1}^{\infty} (-1)^n \frac{4n\pi}{l^2} \sin \frac{2n\pi x}{l} e^{-4n^2\pi^2 t/l^2}.
\]

Obtain another expression for \( u \) by the method of images. Show that for each \( x \), \( 0 \leq x \leq l \),

\[
  \lim_{t \to 0^+} u(x, t) = 0,
\]

and explain why this does not violate the uniqueness principle.

7.11 If \( f(x) \) is a bounded function in \( R_n \), we define its average over \( R_n \) as

\[
  f_{av} = \lim_{r \to \infty} \frac{1}{V_n(r)} \int_{|x| < r} f(x) dx,
\]

where \( V_n(r) \) is the volume of the sphere \( |x| < r \). For heat conduction in \( R_n \) without sources, show that the average of the temperature remains constant in time.

7.12 The equation

\[
  \frac{\partial u}{\partial t} - \nabla^2 u + cu = 0
\]

describes diffusion in which the particles either disintegrate \( (c > 0) \) or multiply \( (c < 0) \) at a rate proportional to the concentration. The one-dimensional version for \( c > 0 \) also describes heat conduction in a rod in which there is radiation from the lateral surface. By an eigenfunction expansion, solve the above equation for a bounded region \( R \), when \( u \) vanishes on the boundary, and \( u(x, 0) = f(x) \).
7.13 If the diffusion coefficient is a function of position \( k(x) \) and if the particles are subject to drift and if there is either disintegration or multiplication proportional to the concentration, then we obtain the rather general parabolic equation

\[
\frac{\partial u}{\partial t} - \text{div} (k \text{ grad } u) + \sum_{i=1}^{n} \alpha_i \frac{\partial u}{\partial x_i} + cu = 0. \tag{7.108}
\]

If \( c = 0 \), and \( k(x) > 0 \), prove the maximum principle (Theorem 2, Section 7.4) for this equation as follows. Let \( M \) be the maximum of the initial and boundary values of \( u(x, t) \). We want to show \( u(x, t) \leq M \) in \( \bar{D} \). Assume the contrary; then there is a point \((x_0, t_0)\), \( x_0 \) in \( R \), \( 0 < t_0 \leq T \), such that \( u(x_0, t_0) = M + \varepsilon \), where \( \varepsilon > 0 \). Then \( \nabla u = 0 \), \( \partial u / \partial t \geq 0 \), and \( \nabla^2 u \leq 0 \) at \((x_0, t_0)\). This still does not contradict the differential equation, since all terms can be 0. Consider

\[
v(x, t) = u(x, t) + \alpha(t_0 - t),
\]

where \( 0 < \alpha < \varepsilon/2T \). Then the initial and boundary values of \( v \) do not exceed \( M + \varepsilon/2 \). Since \( v \) is continuous in \( \bar{D} \), it must assume a maximum at some point \((x_1, t_1)\). Thus

\[
v(x_1, t_1) \geq v(x_0, t_0) = u(x_0, t_0) = M + \varepsilon.
\]

Therefore, \( t_1 > 0 \) and \( x_1 \) is in \( R \). At \((x_1, t_1)\) show that \( \nabla u = 0 \), \( \nabla^2 u \leq 0 \), and \( \partial u / \partial t \geq \alpha > 0 \). This contradicts the differential equation and proves the maximum principle. [Note that \( \text{div} (k \text{ grad } u) = k \nabla^2 u + \text{grad } k \cdot \text{grad } u \).] The minimum principle follows by considering the function \(-u\). If \( c > 0 \), we can only prove that \( u \) cannot have a positive maximum or a negative minimum in the interior.

7.14 Prove Theorem 1, Section 7.4, when the boundary condition is \( \partial u / \partial n = 0 \) or \((\partial u / \partial n) + \theta u = 0 \), where \( \theta > 0 \).

7.15 Prove Theorem 1, Section 7.4, for the differential equation (7.108) of Exercise 7.13, with \( c \geq 0 \).

7.16 (a) Consider the following problem for the heat equation without initial conditions:

\[
\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = q(x)e^{-i\omega t}, \quad 0 < x < \infty, \quad -\infty < t < \infty,
\]

\[
u(0, t) = Ae^{-i\omega t},
\]

where \( \omega \) is a positive real number, \( q(x) \) and \( A \) are given, and \( q(x) \) vanishes for sufficiently large \( x \). By separation of variables show that there exists a bounded solution of the form

\[
u(x, t) = e^{-i\omega t}X(x).
\]
Thus if the sources and the boundary data are periodic in time, so is the solution. In particular, if \( q(x) = 0 \), show that
\[
u(x, t) = Ae^{-i\omega t}e^{(\omega/2)t^2(-1+i)x}\.
\]
By taking the real part of this solution, show that if \( q(x) = 0 \) and \( u(0, t) = A \cos \omega t \),
\[
u(x, t) = Ae^{-(\omega/2)t^2}x \cos \left[ \omega t - \left( \frac{\omega}{2} \right)^{1/2} x \right],
\]
which can be interpreted as a temperature wave traveling with speed \((2\omega)^{1/2}\) and decaying exponentially in \(x\).

(b) Consider the initial value problem
\[
\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < x < \infty, \quad t > 0; \quad u(0, t) = A \cos \omega t.
\]
By taking a Laplace transform on time, show that the solution can be written
\[
u(x, t) = Ae^{-(\omega/2)t^2}x \cos \left[ \omega t - \left( \frac{\omega}{2} \right)^{1/2} x \right]
- \frac{A}{\pi} \int_0^\infty e^{-\rho t} \sin \frac{\rho^{1/2}x}{\rho^2 + \omega^2} \rho d\rho.
\]
Obtain the same formula by taking the Fourier cosine transform on the space coordinate. Find the leading term in the asymptotic expansion of the integral for large \(t\).

7.17 Consider the singular problem for the Bessel equation of order zero with parameter \(\lambda\) in \(a < x < \infty\).
\[
-(xu')' - \lambda xu = 0, \quad a < x < \infty; \quad u(a) = 0.
\]
The point at infinity is a singular point of the limit-point type (see Example 2(b), p. 302, Volume I). To obtain the spectral representation, we construct the Green's function for \(\lambda \neq [0, \infty)\), which satisfies
\[
-(xg')' - \lambda xg = \delta(x - \xi), \quad a < x, \xi < \infty;
\]
\[
g \big|_{x=a} = 0; \quad \int_a^\infty x|g|^2 \, dx < \infty.
\]
If \(\sqrt{\lambda}\) stands for the square root with positive imaginary part, then \(H_0^{(1)}(\sqrt{\lambda} x)\) is the only solution which is of finite norm. Therefore, show that
\[
g(x | \xi; \lambda) = \frac{\pi i}{2H_0^{(1)}(\sqrt{\lambda} a)} H_0^{(1)}(\sqrt{\lambda} x) Z_0(\sqrt{\lambda} x),
\]
where
\[
Z_0(\sqrt{\lambda} x) = J_0(\sqrt{\lambda} x)H_0^{(1)}(\sqrt{\lambda} a) - J_0(\sqrt{\lambda} a)H_0^{(1)}(\sqrt{\lambda} x),
\]
and \(x_+ = \max (x, \xi), x_- = \min (x, \xi)\). Using the formulas
\[
\frac{1}{2\pi i} \int g \; d\lambda = -\frac{\delta(x - \xi)}{x},
\]
\[
H_0^{(1)}(x) - H_0^{(1)}(-x) = 2J_0(x), \quad 0 < x < \infty,
\]
derive the equation
\[
\frac{\delta(x - \xi)}{x} = -\int_0^\infty \mu \; d\mu \; \frac{Z_0(\mu x)Z_0(\mu \xi)}{J_0^2(\mu a) + N_0^2(\mu a)}, \quad a < x, \xi < \infty.
\]
Obtain the Weber transform relations
\[
F_w(\mu) = \int_a^\infty xf(x)Z_0(\mu x)dx,
\]
\[
f(x) = -\int_0^\infty \frac{Z_0(\mu x)F_w(\mu)}{J_0^2(\mu a) + N_0^2(\mu a)} \mu \; d\mu,
\]
where we recall that
\[
Z_0(\mu x) = J_0(\mu x)H_0^{(1)}(\mu a) - J_0(\mu a)H_0^{(1)}(\mu x).
\]
7.18 Obtain (7.76) by inversion of the Laplace transform and also by using the Weber transform of Exercise 7.17.

7.6 PRELIMINARY CONSIDERATIONS FOR THE UNDAMPED WAVE EQUATION

Uniqueness

Consider the general problem (7.5) for the wave equation, repeated here for convenience:
\[
\frac{\partial^2 u}{\partial t^2} - \nabla^2 u = q(x, t), \quad x \text{ in } R, \quad t > 0;
\]
\[
u(x, 0) = f_1(x); \quad \frac{\partial u}{\partial t} (x, 0) = f_2(x), \quad x \text{ in } R, \quad (7.109)
\]
\[
u(x, t) = h(x, t), \quad x \text{ on } \sigma, \quad t > 0.
\]
For simplicity we shall assume (although it is not necessary to do so) that \(q, f_1, f_2, h, \) and \(u\) are real. Moreover, to ensure continuity of \(u\) at \(t = 0,\) we shall take the boundary value of \(u\) to be compatible with the initial value of \(u;\) that is, \(h(x, 0+) = f_1(x),\) \(x \text{ on } \sigma.\)
Referring again to Figure 7.1, we shall prove that (7.109) has at most one continuous solution in the finite cylinder $\bar{D}$. If two solutions existed, their difference $v(x, t)$ would be continuous in $\bar{D}$ and would satisfy the homogeneous system

$$\frac{\partial^2 v}{\partial t^2} - \nabla^2 v = 0, \quad x \text{ in } R, \quad 0 < t < T;$$

$$v(x, 0) = \frac{\partial v}{\partial t}(x, 0) = 0, \quad x \text{ in } R; \quad (7.110)$$

$$v(x, t) = 0, \quad x \text{ on } \sigma, \quad 0 < t < T.$$

We multiply the differential equation for $v$ by $\partial v/\partial t$ to obtain

$$\frac{1}{2} \frac{\partial}{\partial t} \left[ \left( \frac{\partial v}{\partial t} \right)^2 \right] - \text{div} \left( \frac{\partial v}{\partial t} \text{ grad } v \right) + \frac{1}{2} \frac{\partial}{\partial t} |\text{grad } v|^2 = 0.$$

Integration over $R$ yields

$$\frac{1}{2} \frac{d}{dt} \int_R \left[ \left( \frac{\partial v}{\partial t} \right)^2 + |\text{grad } v|^2 \right] dx = \int_{\sigma} dS \frac{\partial v}{\partial t} \frac{\partial v}{\partial n},$$

which can be interpreted as describing conservation of energy. The first term on the left is the rate of change of kinetic energy, the second, the rate of change of strain energy. The term on the right is the work done by the boundary forces. In our case, $v$ vanishes on $\sigma$, so that $\partial v/\partial t = 0$ on $\sigma$; hence

$$\frac{1}{2} \int_R \left[ \left( \frac{\partial v}{\partial t} \right)^2 + |\text{grad } v|^2 \right] dx = C, \quad 0 < t < T,$$

where $C$ is a constant. At $t = 0$, $\partial v/\partial t = 0$, and, since $v = 0$, grad $v = 0$. Therefore $C = 0$, and

$$\left( \frac{\partial v}{\partial t} \right)^2 + |\text{grad } v|^2 = 0, \quad 0 < t < T, \quad x \text{ in } R.$$

This implies that $v$ is constant, and again, since $v$ vanishes at $t = 0$, we find $v \equiv 0$, $x \text{ in } R, 0 < t < T$, which completes the proof.

It should be pointed out that the proof can be carried out whenever

$$\int_{\sigma} dS \frac{\partial v}{\partial t} \frac{\partial v}{\partial n} \leq 0.$$

This will be the case if $\partial u/\partial n$ or $\partial u/\partial n + \theta u$, with $\theta$ positive, is assigned in (7.109) instead of $u$.  

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Next we obtain a useful result relating the solutions of the two initial value problems,

\[
\frac{\partial^2 u}{\partial t^2} - \nabla^2 u = 0, \quad x \text{ in } R, \quad t > 0;
\]

\[
u(x, 0) = 0; \quad \frac{\partial u}{\partial t} (x, 0) = f(x), \quad x \text{ in } R; \tag{7.111}\]

\[
u(x, t) = 0, \quad x \text{ on } \sigma, \quad t > 0.
\]

\[
\frac{\partial^2 v}{\partial t^2} - \nabla^2 v = 0, \quad x \text{ in } R, \quad t > 0;
\]

\[
v(x, 0) = f(x); \quad \frac{\partial v}{\partial t} (x, 0) = 0, \quad x \text{ in } R; \tag{7.112}\]

\[
v(x, t) = 0, \quad x \text{ on } \sigma, \quad t > 0.
\]

These problems differ only in the initial conditions—for the first problem the displacement is 0 and the velocity \(f(x)\), whereas for the second problem the velocity is 0 and the displacement is \(f(x)\).

**Theorem**

\[
v(x, t) = \frac{\partial u}{\partial t}.
\]

**Proof.** Let \(u(x, t)\) be the solution of (7.111) and let \(w(x, t)\) be defined as

\[
w(x, t) = \frac{\partial u}{\partial t}.
\]

We must show that \(w\) satisfies (7.112). Clearly \(w\) is a solution of the homogeneous wave equation, since

\[
\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)w = \left(\frac{\partial^2}{\partial t^2} - \nabla^2\right) \frac{\partial u}{\partial t} = \frac{\partial}{\partial t} \left(\frac{\partial^2}{\partial t^2} - \nabla^2\right) u = 0.
\]

Since \(u\) vanishes on \(\sigma\) for all \(t > 0\), so does \(\partial u/\partial t\), hence \(w\). Also

\[
w(x, 0) = \frac{\partial u}{\partial t} (x, 0) = f(x),
\]

\[
\frac{\partial w}{\partial t} (x, t) = \frac{\partial^2 u}{\partial t^2} (x, t) = \nabla^2 u, \quad t > 0,
\]

so that

\[
\lim_{t \to 0^+} \frac{\partial w}{\partial t} (x, t) = \frac{\partial w}{\partial t} (x, 0) = \lim_{t \to 0^+} \nabla^2 u(x, t) = \nabla^2 u(x, 0) = 0,
\]

and we have shown that \(w\) satisfies (7.112).
The theorem permits us to concentrate our attention on boundary value problems of the type (7.111) with zero initial displacement. By the theorem we can then solve (7.112) and, by superposition, we can also handle the problem with arbitrary initial displacement and velocity.

### 7.7 CAUSAL GREEN’S FUNCTION FOR THE WAVE EQUATION

The causal Green’s function \( g(x, t \mid x_0, t_0) \) for the wave equation in a space region \( R \), with vanishing boundary condition on the boundary \( \sigma \) of \( R \), satisfies

\[
\left( \frac{\partial^2}{\partial t^2} - \nabla^2 \right) g(x, t \mid x_0, t_0) = \delta(x - x_0)\delta(t - t_0), \quad -\infty < t, t_0 < \infty, \quad x, x_0 \in R; \\
g \equiv 0, \quad t < t_0; \\
g = 0, \quad x \text{ on } \sigma.
\]  

(7.113)

If the region \( R \) is unbounded, a boundary condition at \( |x| = \infty \) may also be needed.

Since the equation is invariant under time translation, it is easy to see that

\[
g(x, t \mid x_0, t_0) = g(x, t - t_0 \mid x_0, 0),
\]

so that it will suffice to find \( g \) for a source introduced at \( t = 0 \).

By the same argument used in going from (5.142) to (5.145), \( g(x, t \mid x_0, t_0) \) is seen to be equally well characterized for \( t > t_0 \) by the initial value problem

\[
\left( \frac{\partial^2}{\partial t^2} - \nabla^2 \right) g(x, t \mid x_0, t_0) = 0, \quad t > t_0, \quad x, x_0 \in R; \\
\lim_{t \to t_0^+} g(x, t \mid x_0, t_0) = 0; \quad \lim_{t \to t_0^+} \frac{\partial g}{\partial t} = \delta(x - x_0); \\
g = 0, \quad x \text{ on } \sigma, \quad t > t_0.
\]  

(7.114)

Thus the source term \( \delta(x - x_0)\delta(t - t_0) \) appearing in (7.113) is equivalent to an initial velocity \( \delta(x - x_0) \) at time \( t_0 \) in (7.114). The physical interpretation for the case of a membrane is very simple. If the membrane is at rest until time \( t_0 \), when an impulsive, concentrated force is applied at \( x_0 \), there is no immediate change in the deflection but the velocity is instantaneously raised from 0 to \( \delta(x - x_0) \). Henceforth we shall freely use either of the characterizations of \( g \) as convenience or inclination dictates.

For the heat equation we have seen that \( g \) is infinitely differentiable for \( t > t_0 \); quite the contrary is true for the wave equation, where \( g \) will usually not be continuous and in fact will often be a distribution of the Dirac type rather than a function. Thus (7.113) or (7.114) should be interpreted in the generalized sense. This was done with some care in Chapter 5, and we shall not dwell here on the details but rather proceed in a purely formal manner.
If we interchange the source and observation point in \( g \), we find that
\[
\left( \frac{\partial^2}{\partial t^2} - \nabla^2 \right) g(x_0, t_0 | x, t) = \delta(x - x_0)\delta(t - t_0),
\]
\[-\infty < t, t_0 < \infty, \quad x, x_0 \text{ in } \mathbb{R};
\]
\[g(x_0, t_0 | x, t) = 0, \quad t > t_0; \tag{7.115}
\]
g = 0, \quad x \text{ on } \sigma.

To solve (7.109) we apply Green's theorem (7.7) to the finite cylinder \( D \) of Figure 7.1, with \( u \) the solution of (7.109) and \( v = g(x_0, t_0 | x, t) \). Here \( T \) is any value of time greater than \( t_0 \). Using the fact that \( g(x_0, t_0 | x, t) \) vanishes for \( t > t_0 \), and that \( \partial g/\partial t = -\partial g/\partial t_0 \), we obtain
\[
u(x_0, t_0) = \int_0^{t_0} \int_R dx \ g(x_0, t_0 | x, t)q(x, t) + \int_R dx \ g(x_0, t_0 | x, 0)f_2(x)
\]
\[+ \frac{\partial}{\partial t_0} \left[ \int_R dx \ g(x_0, t_0 | x, 0)f_1(x) \right]
\]
\[- \int_0^{t_0} dt \int_\sigma dS_x h(x, t) \frac{\partial g}{\partial n_x}(x_0, t_0 | x, t). \tag{7.116}
\]

Note that the third term, which gives the dependence on the initial value of \( u \), is related to the second term (giving dependence on initial velocity) exactly as predicted by the theorem of Section 7.6.

Since \( T \) does not appear in (7.116), the result is valid for all \( t_0 \). Moreover, it is clear from the formula that the solution at time \( t_0 \) does not depend on sources or boundary values applied after time \( t_0 \).

All the methods used in calculating the Green's function for the heat equation apply equally well for the wave equation. One of these methods involves an expansion in the space eigenfunctions \( \{\varphi_k\} \) satisfying
\[-\nabla^2 \varphi_k = \lambda_k \varphi_k, \quad x \text{ in } \mathbb{R}; \quad \varphi_k = 0, \quad x \text{ on } \sigma. \tag{7.117}\]

Writing
\[
g(x, t | x_0, t_0) = \sum_{k=1}^{\infty} g_k \varphi_k(x),
\]
\[g_k = \int_R g \bar{\varphi}_k(x)dx,
\]
we obtain, by multiplying (7.114) by \( \bar{\varphi}_k(x) \) and integrating over \( R \),
\[
\frac{d^2}{dt^2} \ g_k - \int_R \bar{\varphi}_k(x) \nabla^2 g \ dx = 0.
\]
Integrating by parts and using (7.117) and the boundary condition on \( g \), we find
\[
\frac{d^2 g_k}{dt^2} + \lambda_k g_k = 0, \quad t > t_0; \quad g_k \bigg|_{t=t_0^+} = 0, \quad \frac{dg_k}{dt} \bigg|_{t=t_0^+} = \bar{\varphi}_k(x_0),
\]
\[
g_k = \bar{\varphi}_k(x_0) \frac{\sin \sqrt{\lambda_k}(t - t_0)}{\sqrt{\lambda_k}}.
\]

Therefore we have the bilinear series for \( g \):
\[
g(x, t \mid x_0, t_0) = \sum_{k=1}^{\infty} \frac{\sin \sqrt{\lambda_k}(t - t_0)}{\sqrt{\lambda_k}} \varphi_k(x) \bar{\varphi}_k(x_0).
\] (7.118)

For the case of a bounded region \( R \), this expression for \( g \) can be used as it stands, but, for an unbounded region, the spectrum of (7.117) is continuous and the summation in (7.118) becomes an integral, and it is then usually preferable to apply the appropriate space transform directly on (7.114).

Substituting (7.118) in (7.116) we obtain the space eigenfunction expansion of the solution of (7.109).

Of particular interest is the case of free vibrations, where \( g(x, t) \) and \( h(x, t) \) are 0. Then we find
\[
u(x, t) = \sum_{k=1}^{\infty} f_{1,k} \frac{\sin \sqrt{\lambda_k} t}{\sqrt{\lambda_k}} \varphi_k(x) + \sum_{k=1}^{\infty} f_{2,k} (\cos \sqrt{\lambda_k} t) \varphi_k(x),
\]
where \( f_{1,k} \) and \( f_{2,k} \) are the respective Fourier coefficients of \( f_1 \) and \( f_2 \) in the orthonormal set \( \{ \varphi_k \} \). Clearly it is possible to adjust the initial conditions so that the solution consists of a single term of the form
\[
[a_k \sin \sqrt{\lambda_k} t + b_k \cos \sqrt{\lambda_k} t] \varphi_k(x).
\]

Such a deflection is then always of the same shape \( \varphi_k(x) \) with an amplitude which varies periodically in \( t \) (with frequency \( \sqrt{\lambda_k} \)). For this reason we say that \( \varphi_k(x) \) is a natural mode with natural frequency \( \sqrt{\lambda_k} \). It should be observed that the deflection \( u(x, t) \) corresponding to an arbitrary set of initial values is not usually periodic in time. Only if the eigenvalues \( \sqrt{\lambda_k} \) are all integer multiples of some fundamental frequency will the general motion be periodic. This would imply that \( \lambda_k = \alpha k^2 \), for all \( k \), which can happen only for the simplest one-dimensional problems.

Alternatively, we may find \( g \) by taking a Laplace transform on time (expansion in the time "eigenfunctions" \( e^{st} \)). For brevity of notation we suppress the dependence in \( x_0 \), take \( t_0 = 0 \), and write
\[
g(x, t \mid x_0, 0) = g(x, t).
\]
The Laplace transform of \( g \) is

\[
\tilde{g}(x, s) = \int_0^\infty e^{-st}g(x, t)dt,
\]

and, multiplying the differential equation in (7.114) by \( e^{st} \), setting \( t_0 = 0 \), integrating from \( t = 0 \) to \( \infty \), we find

\[
-\nabla^2 \tilde{g} + s^2 \tilde{g} = \delta(x - x_0), \quad x, x_0 \text{ in } R; \quad \tilde{g} = 0, \quad x \text{ on } \sigma.
\]

The problem is then reduced to finding the Green's function for the Helmholtz equation for the space region \( R \). If \( R \) is bounded, we may expand \( \tilde{g} \) in the eigenfunctions of (7.117), with the result

\[
\tilde{g} = \sum_{k=1}^\infty \frac{\varphi_k(x_0)}{s^2 + \lambda_k},
\]

with

\[
g(x, t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \tilde{g} e^{st} ds, \quad a > 0.
\]

Performing the inversion by closing the contour to the left in the \( s \) plane, we pick up residues at the simple poles, \( s = \pm i\sqrt{\lambda_k} \). An easy calculation gives us once more the formula (7.118) with \( t_0 = 0 \).

### 7.8 PROBLEMS IN ONE SPACE DIMENSION

#### Infinite String

Consider first the causal fundamental solution for an infinite string \(-\infty < x < \infty\). By applying a Fourier transform on space or a Laplace transform on time we find, as in (5.150),

\[
C_1(x, t | x_0, t_0) = \frac{1}{2}H(t - t_0 - |x - x_0|),
\]

(7.119)

where we have used the letter \( C_1 \) instead of \( g \) to indicate that we are dealing with the entire one-dimensional space. The solution (7.119) corresponds to a string which is at rest up to \( t = t_0 \) when a blow is struck at \( x = x_0 \). A well forms at \( x = x_0 \) whose front spreads out with velocity one (see Figure 7.7). The velocity of propagation of the front being unity is a consequence of having set \( c^2 = 1 \) in the wave equation. If we returned to the case of a general \( c^2 \), we would find that the velocity of propagation is \( c \).

If instead of striking the string at the point \( x_0 \) at time \( t_0 \), we pluck the string, that is, give it no initial velocity but an initial deflection \( \delta(x - x_0) \), then by the theorem of Section 7.6, the corresponding deflection is

\[
u(x, t | x_0, t_0) = \frac{\partial C_1}{\partial t} = \frac{1}{2}\delta(t - t_0 - |x - x_0|).
\]
Thus the initial deflection splits into two pulses, of half the original size, traveling to the right and to the left, respectively, with velocity 1.

For $t > t_0$, we can rewrite (7.119) as

$$C_1(x, t | x_0, t_0) = \frac{1}{2} \{ H[t - t_0 - (x - x_0)] \\
+ H[t - t_0 + (x - x_0)] - 1 \}, \quad t > t_0. \quad (7.120)$$

Note that each of the terms in this expression is a solution of the homogeneous wave equation for all $t$. This does not contradict the fact that $C_1$ satisfies (7.113) with a delta function as a source term, since the formula (7.120) does not represent $C_1$ for $t < t_0$ (in fact, $C_1 \equiv 0$ for $t < t_0$).

Consider next the general initial value problem for an infinite string,

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = q(x, t), \quad t > 0, \quad -\infty < x < \infty; \quad (7.121)$$

$$u(x, 0) = f_1(x), \quad \frac{\partial u}{\partial t}(x, 0) = f_2(x).$$

Since

$$C_1(x_0, t_0 | x, t) = \frac{1}{2} H(t_0 - t - |x - x_0|),$$

$$\frac{\partial C_1}{\partial t_0}(x_0, t_0 | x, t) = \frac{1}{2} \delta(t_0 - t - |x - x_0|),$$

we find from (7.116) that

$$u(x_0, t_0) = \int_0^{t_0} dt \int_{x_0 - (t_0 - t)}^{x_0 + (t_0 - t)} dx \frac{1}{2} q(x, t) + \frac{1}{2} [f_1(x_0 - t_0) + f_1(x_0 + t_0)]$$

$$+ \frac{1}{2} \int_{x_0 - t_0}^{x_0 + t_0} f_2(x) dx. \quad (7.122)$$
From Figure 7.8 we see that only those sources \( q \) in the shaded region affect the solution at \((x_0, t_0)\). Only the initial values of the velocity between \(A\) and \(B\) and the initial values of the deflection at \(A\) and \(B\) affect the solution at \((x_0, t_0)\). Here \(A\) and \(B\) have the coordinates \((x_0 - t_0, 0)\) and \((x_0 + t_0, 0)\), respectively. One can easily establish from (7.122) that the solution depends continuously on the data.

**Semiinfinite String**

The causal Green's function for the semiinfinite string \(0 < x < \infty\) with vanishing deflection at \(x = 0\) satisfies

\[
\frac{\partial^2 g}{\partial t^2} - \frac{\partial^2 g}{\partial x^2} = \delta(x - x_0)\delta(t - t_0), \quad 0 < x, x_0 < \infty, \quad -\infty < t, t_0 < \infty;
\]

\[
g(x, t | x_0, t_0) = 0, \quad t < t_0;
\]

\[
g(0, t | x_0, t_0) = 0.
\]

We can easily calculate \(g\) from \(C_1\) by images. Placing an additional source of opposite sign at the point \(-x_0\) in an infinite string will give us just what we want for the semiinfinite string, the odd symmetry of the sources about \(x = 0\) guaranteeing that the deflection will remain 0 at \(x = 0\) for all \(t\). Thus

\[
g(x, t | x_0, t_0) = C_1(x, t | x_0, t_0) - C_1(x, t | -x_0, t_0). \quad (7.123)
\]

The only novelty in applying (7.116) is the presence of a boundary term corresponding to the space boundary \(x = 0\). We therefore only analyze that part of the problem for a semiinfinite string. Consider then the boundary value problem

\[
\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0, \quad x > 0, \quad t > 0; \quad u(x, 0) = \frac{\partial u}{\partial t} (x, 0) = 0; \quad u(0, t) = h(t). \quad (7.124)
\]
Formula (7.116) requires the preliminary calculation of
\[
\frac{\partial g}{\partial n_x}(x_0, t_0 \mid x, t) = -\frac{\partial g}{\partial x}(x_0, t_0 \mid x, t)\bigg|_{x=0}.
\]

For \(t_0 > t\), we have
\[
g(x_0, t_0 \mid x, t) = \frac{1}{2}\{H[t_0 - t - (x_0 - x)] + H[t_0 - t + (x_0 - x)]
- H[t_0 - t - (x_0 + x)] - H[t_0 - t + (x_0 + x)]\},
\]
\[
\frac{\partial g}{\partial x}\bigg|_{x=0} = \delta(t_0 - t - x_0) - \delta(t_0 - t + x_0), \quad t_0 > t.
\]

Since \(x_0 > 0\) and \(t_0 > t\), the second delta function vanishes. Therefore (7.116) becomes
\[
u(x_0, t_0) = \int_0^t \delta(t_0 - t - x_0)h(t)dt = \begin{cases} h(t_0 - x_0), & 0 < x_0 < t_0; \\ 0, & t_0 < x_0. \end{cases}
\]

Thus the boundary deflection propagates with velocity 1 along the string. Observe that even though \(u\) is discontinuous, it still satisfies the homogeneous wave equation in the generalized sense of Chapter 5. The present result was also obtained earlier by the method of characteristics and can also be derived by a sine transform on the space coordinate or a Laplace transform on time (see Exercise 7.26).

**Finite String**

The causal Green's function \(g(x, t \mid x_0, t_0)\) for a string fixed at its ends \(x = 0\) and \(x = l\) can be obtained from the causal fundamental solution (7.119) by the method of images, in the same way as for the heat equation. We consider, for an infinite string, a system consisting of unit positive sources at \(x = x_0 + 2nl\) and of unit negative sources at \(x = -x_0 + 2nl\). The deflection for this infinite string will coincide in the interval \(0 < x < l\) with the desired solution for the finite string. Thus
\[
g(x, t \mid x_0, t_0) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \{H(t - t_0 - |x - x_0 - 2nl|) - H(t - t_0 - |x + x_0 - 2nl|)\},
\]
and we can easily interpret this solution in terms of successive reflections from the ends of the string. From (7.118) we find the alternative expression
\[
g(x, t \mid x_0, t_0) = \sum_{k=1}^{\infty} \frac{2}{k\pi} \sin \frac{k\pi(t - t_0)}{l} \sin \frac{k\pi}{l} \sin \frac{k\pi x_0}{l},
\]

(7.126)
since the eigenfunctions of (7.117) are \((2/l)^{1/2} \sin (k\pi x/l)\) and the eigenvalues \(k^2\pi^2/l^2\). As an example of a simple initial value problem for the finite string, consider
\[
\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0, \quad u(x, 0) = f(x); \quad \frac{\partial u}{\partial t}(x, 0) = 0, \quad u(0, t) = u(l, t) = 0.
\]
Either by (7.116) and (7.126) or by direct separation of variables, we find
\[
u(x, t) = \sum_{k=1}^{\infty} f_k \cos \frac{k\pi t}{l} \sin \frac{k\pi x}{l},
\]
where
\[
f_k = \frac{2}{l} \int_0^l f(x) \sin \frac{k\pi x}{l} \, dx,
\]
\[
f(x) = \sum_{k=1}^{\infty} f_k \sin \frac{k\pi x}{l}, \quad 0 < x < l.
\]
The expression for \(u\) can be rewritten
\[
u(x, t) = \frac{1}{2} \sum_{k=1}^{\infty} f_k \left[ \sin \frac{k\pi}{l} (x + t) + \sin \frac{k\pi}{l} (x - t) \right],
\]
or
\[
u(x, t) = \frac{1}{2} [f^*(x + t) + f^*(x - t)],
\]
where \(f^*(x)\) is a function defined from \(-\infty\) to \(\infty\) and
\[
f^*(x) = \sum_{k=1}^{\infty} f_k \sin \frac{k\pi}{l} x.
\]
Thus \(f^*\) coincides with \(f\) in \(0 < x < l\); in \(-l < x < 0\), \(f^*\) is the odd extension of \(f\); outside \((-l, l)\), \(f^*\) is the periodic extension, with period \(2l\), of the values already defined in \((-l, l)\). For brevity we call \(f^*\) the odd periodic extension of \(f\).

Similar considerations also apply to the finite string problem with fixed ends and arbitrary initial velocity or arbitrary sources. Thus we merely have to construct the odd periodic extensions of the initial displacement, velocity, and source term, and then solve the corresponding problem for the infinite string. The part of this solution lying between \(x = 0\) and \(x = l\) is then the solution of the finite string. If the boundary condition at \(x = 0\) and \(x = l\) is that \(\partial u/\partial x\) vanishes, we must consider instead the even periodic extensions of the data.

### 7.9 PROBLEMS IN MORE THAN ONE DIMENSION

The causal fundamental solution for the wave equation when the space region is the whole of three-dimensional space was calculated in (5.152) as
\[
C_3(x, t \mid x_0, t_0) = \frac{1}{4\pi|x - x_0|} \delta(t - t_0 - |x - x_0|). \quad (7.127)
\]
This causal solution can be used to solve the general initial value problem for the wave equation in three space dimensions:

\[
\frac{\partial^2 u}{\partial t^2} - \nabla^2 u = q(x, t); \quad x \text{ in } R_3, \quad t > 0; \quad u(x, 0) = f_1(x); \quad \frac{\partial u}{\partial t}(x, 0) = f_2(x).
\]

(7.128)

In fact, from (7.116), we have

\[
u(x_0, t_0) = I + J + K,
\]

where

\[
I = \int_0^{t_0} dt \int_{R_3} dx \frac{\delta(t_0 - t - |x_0 - x|)}{4\pi |x_0 - x|} q(x, t),
\]

\[
J = \int_{R_3} dx \frac{\delta(t_0 - |x_0 - x|)}{4\pi |x_0 - x|} f_2(x),
\]

\[
K = \frac{\partial}{\partial t_0} \int_{R_3} dx \frac{\delta(t_0 - |x_0 - x|)}{4\pi |x_0 - x|} f_1(x).
\]

The explicit calculations are simple. Consider \( I \) and perform the integration with respect to \( t \) first; then

\[
\int_0^{t_0} dt \delta(t_0 - t - |x_0 - x|) q(x, t) = \begin{cases} 
q(x, t_0 - |x_0 - x|), & t_0 > |x_0 - x|, \\
0, & t_0 < |x_0 - x|,
\end{cases}
\]

\[
I(x_0, t_0) = \int_{|x - x_0| < t_0} dx \frac{q(x, t_0 - |x_0 - x|)}{4\pi |x_0 - x|}.
\]

Thus if we think of disturbances traveling with unit velocity, \( I \) is the sum of all the source disturbances which are \textit{just} reaching \( x_0 \) at time \( t_0 \), that is, the disturbances from sources which acted at any point \( x \) at time \( t_0 - |x - x_0| \). From this point of view, it is of course understood that all sources \( q(x, t) \) are 0 for \( t < 0 \). The quantity

\[
\frac{q(x, t_0 - |x_0 - x|)}{4\pi |x_0 - x|}
\]

is known as the \textit{retarded potential}.

To calculate \( J \), let us introduce spherical coordinates \((r, \theta, \varphi)\) with origin at the fixed observation point \( x_0 \). Then

\[
J = \int_0^\pi d\theta \int_0^{2\pi} d\varphi \int_0^\infty dr \ r^2 \sin \theta \ \frac{\delta(t_0 - r)}{4\pi r} f_2(r, \theta, \varphi),
\]

\[
= t_0 \left[ \frac{1}{4\pi} \int_0^\pi d\theta \int_0^{2\pi} d\varphi \sin \theta f_2(t_0, \theta, \varphi) \right].
\]
The term in brackets is the average value of \( f_2 \) on the surface of a sphere of radius \( t_0 \) with center at \( x_0 \). We denote this average by

\[
f_2^*(x_0, t_0).
\]

Therefore, only those initial velocities contributing disturbances which just reach the point \( x_0 \) at time \( t_0 \) affect the solution at \((x_0, t_0)\). Hence

\[
J = t_0 f_2^*(x_0, t_0),
\]

\[
K = \frac{\partial}{\partial t_0} [t_0 f_1^*(x_0, t_0)] = t_0 \frac{\partial}{\partial t_0} [f_1^*(x_0, t_0)] + f_1^*(x_0, t_0).
\]

Finally,

\[
u(x_0, t_0) = \int_{|x-x_0|<t_0} dx \frac{q(x, t_0 - |x_0 - x|)}{4\pi|x_0 - x|}
\]

\[
+ t_0 f_2^*(x_0, t_0) + \frac{\partial}{\partial t_0} [t_0 f_1^*(x_0, t_0)],
\]

(7.129)

a result due to Poisson.

In Chapter 5, equation (5.151), we also calculated the two-dimensional causal solution \( C_2 \), but we now give another derivation adapted from Hadamard’s method of descent. The idea is to write the source term for \( C_2 \) (which when viewed in three space dimensions is a line source) as a superposition of three-dimensional point sources. With Cartesian coordinates \( x, y, z \), consider a line source of unit line density along the \( z \) axis, the source being instantaneously released at \( t = 0 \). The source function is then \( \delta(x)\delta(y)\delta(t) \) and the response will obviously be independent of \( z \) and clearly coincides with \( C_2(x, y, t | 0, 0, 0) \), which we abbreviate \( C_2(x, y, t) \). The line source can be divided into individual elements of length \( dz_0 \), each carrying a concentrated point source of strength \( dz_0 \). By (7.127) the element located between \((0, 0, z_0)\) and \((0, 0, z_0 + dz_0)\) gives rise at \((x, y, 0)\) to the response

\[
\frac{1}{4\pi(x^2 + y^2 + z_0^2)^{1/2}} \delta[t - (x^2 + y^2 + z_0^2)^{1/2}].
\]

Setting \( x^2 + y^2 = \rho^2 \) and adding all these elementary contributions, we find

\[
C_2(x, y, t) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{1}{(\rho^2 + z_0^2)^{1/2}} \delta[t - (\rho^2 + z_0^2)^{1/2}]dz_0
\]

\[
= \frac{1}{2\pi} \int_{0}^{\infty} \frac{1}{(\rho^2 + z_0^2)^{1/2}} \delta[t - (\rho^2 + z_0^2)^{1/2}]dz_0.
\]

If \( t < \rho \), the argument of the delta function in the last integral never vanishes so that \( C_2 = 0 \); if \( t > \rho \), we make the change of variables

\[
\alpha = (\rho^2 + z_0^2)^{1/2}, \quad z_0^2 = \alpha^2 - \rho^2, \quad dz_0 = \frac{\alpha d\alpha}{z_0}.
\]

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to obtain
\[
C_2(x, y, t) = \frac{1}{2\pi} \int_\rho^\infty \frac{\delta(t - \alpha)}{[\alpha^2 - \rho^2]^{1/2}} \, d\alpha = \frac{1}{2\pi} [t^2 - \rho^2]^{-1/2}, \quad t > \rho, \tag{7.130}
\]
which confirms (5.151).

As has already been observed earlier, there is a striking difference between the causal solution in two and three dimensions. For the latter, (7.127) shows that the entire disturbance or response is concentrated on the surface of a sphere traveling outward with velocity 1 from the source. Thus the wave front is sharp and there is no diffusion (or wake); we say that the wave front satisfies Huyghens' principle. Another aspect of this principle shows up in the solution (7.129) of the initial value problem with \( q = 0 \)—the solution at \( x_0, t_0 \) depends only on the initial data on the surface of the sphere \( |x - x_0| = t_0 \) and not on the initial data interior to the sphere. Huyghens' principle applies to all odd space dimensions greater than or equal to three and almost (but not quite) applies for problems in one space dimension. From (7.119) we see that \( C_1(x, t \mid x_0, 0) \) is not concentrated on \( |x - x_0| = t_0 \) but that \( \partial C_1/\partial t \) is. Thus, from (7.122), initial velocities diffuse but initial deflections propagate sharply.

In an even number of space dimensions, there is always a wake both for initial velocities and initial deflections. From (7.130), for instance, it is clear that the wave front is a circle propagating outward with velocity 1 but that there is a residual disturbance after the wave front has passed.

The method of descent can also be applied to obtain \( C_1 \) from \( C_3 \). We can regard the source term for \( C_1(x, t \mid 0, 0) \) as a sheet of sources of unit surface density on the plane \( x = 0 \) \((-\infty < y_0 < \infty, -\infty < z_0 < \infty)\). Let us set \( y_0^2 + z_0^2 = p^2 \); then the annulus between \( p \) and \( p + dp \) carries a source of strength \( 2\pi p \, dp \) which contributes at \( (x, 0, 0) \) a response
\[
2\pi p \, dp \frac{\delta[t - (p^2 + x^2)^{1/2}]}{4\pi(p^2 + x^2)^{1/2}}.
\]
We obtain \( C_1 \) by summing these contributions; hence
\[
C_1(x, t \mid 0, 0) = \frac{1}{2} \int_0^\infty p \, dp \frac{\delta[t - (p^2 + x^2)^{1/2}]}{(p^2 + x^2)^{1/2}}.
\]
If \( t < |x| \), the argument of the delta function does not vanish in the interval of integration and \( C_1 = 0 \). If \( t > |x| \), let
\[
\alpha = (p^2 + x^2)^{1/2}, \quad d\alpha = \frac{p \, dp}{(p^2 + x^2)^{1/2}},
\]
so that
\[
C_1(x, t \mid 0, 0) = \frac{1}{2}, \quad t > |x|,
\]
which agrees with (7.119).
7.10 WAVE EQUATION WITH EXTERNAL DAMPING

Consider the transverse vibrations of a membrane in a surrounding medium (such as air) furnishing a resistance to the motion that is proportional to the velocity. We have seen in Appendix A, equation (A.5), Volume I, that the deflection is governed by an equation of the type

\[ \frac{\partial^2 u}{\partial t^2} + 2\gamma \frac{\partial u}{\partial t} - \nabla^2 u = q(x, t), \]

where \( \gamma \) is a positive constant [see also (5.157)]. In Chapter 5 we discussed the causal fundamental solution of this equation for free space. Here we shall only investigate some special aspects of the equation which have bearing on the wave equation without damping; we shall be particularly concerned therefore with the limit \( \gamma \to 0 \).

We first look at the homogeneous equation in a bounded region \( R \) when \( u \) vanishes on \( \sigma \) (the boundary of \( R \)).

Let \( u(x, t) \) be the solution of

\[ \frac{\partial^2 u}{\partial t^2} + 2\gamma \frac{\partial u}{\partial t} - \nabla^2 u = 0, \quad x \text{ in } R, \quad t > 0; \]

\[ u(x, t) = 0, \quad x \text{ on } \sigma, \quad t > 0; \]

\[ u(x, 0) = 0; \quad \frac{\partial u}{\partial t} (x, 0) = f(x), \quad x \text{ in } R \]  \hspace{1cm} (7.131)

and let \( v(x, t) \) be the solution of

\[ \frac{\partial^2 v}{\partial t^2} + 2\gamma \frac{\partial v}{\partial t} - \nabla^2 v = 0, \quad x \text{ in } R, \quad t > 0; \]

\[ v(x, t) = 0, \quad x \text{ on } \sigma, \quad t > 0; \]

\[ v(x, 0) = f(x); \quad \frac{\partial v}{\partial t} (x, 0) = 0, \quad x \text{ in } R. \] \hspace{1cm} (7.132)

Just as for the undamped wave equations (see Section 7.6) we expect a simple relation between these problems.

**Theorem**

\[ v(x, t) = \frac{\partial u}{\partial t} + 2\gamma u. \] \hspace{1cm} (7.133)

**Proof.** Let \( u \) be the solution of (7.131) and define \( v \) by (7.133). We must show that \( v \) so defined satisfies the system (7.132). Clearly \( v \) satisfies the differential equation and vanishes on \( \sigma \). Also \( v(x, 0) = \frac{\partial u(x, 0)}{\partial t} + 2\gamma u(x, 0) = f(x) \). Now

\[ \frac{\partial v}{\partial t} (x, t) = \frac{\partial^2 u}{\partial t^2} + 2\gamma \frac{\partial u}{\partial t} = \nabla^2 u, \]
and, as $t \to 0$,
\[
\frac{\partial v}{\partial t}(x, 0) = \nabla^2 u(x, 0) = 0.
\]

The advantage of this theorem is that we can restrict ourselves to initial value problems of type (7.131). By using the theorem and superposition we can then handle the problem in which both $u(x, 0)$ and $\partial u(x, 0)/\partial t$ are arbitrary functions.

Let us now find the solution of (7.131). As in Chapter 5, equation (5.157), we make the preliminary transformation
\[
u = we^{-\gamma t},
\]
(7.134)
where $w(x, t)$ satisfies the equation of telegraphy,
\[
\frac{\partial^2 w}{\partial t^2} - \gamma^2 w - \nabla^2 w = 0, \quad x \text{ in } R, \quad t > 0;
\]
\[
w(x, 0) = 0; \quad \frac{\partial w}{\partial t}(x, 0) = f(x); \quad w(x, t) = 0, \quad x \text{ on } \sigma, \quad t > 0.
\]
(7.135)

We now expand $w(x, t)$ in the space eigenfunctions of (7.117). Letting
\[
w(x, t) = \sum_{k=1}^{\infty} w_k(t)\phi_k(x), \quad w_k(t) = \int_R w(x, t)\bar{\phi}_k(x)dx,
\]
multiplying the differential equation in (7.135) by $\bar{\phi}_k(x)$, and integrating over $R$, we find that $w_k(t)$ satisfies the ordinary differential equation
\[
\frac{d^2 w_k}{dt^2} - (\gamma^2 - \lambda_k)w_k = 0, \quad t > 0; \quad w_k(0) = 0, \quad w_k'(0) = f_k,
\]
(7.136)
where
\[
f_k = \int_R f(x)\bar{\phi}_k(x)dx.
\]

Therefore
\[
w_k = \begin{cases}
    f_k \frac{\sinh (\gamma^2 - \lambda_k)^{1/2}t}{(\gamma^2 - \lambda_k)^{1/2}} & \text{for } k \text{ such that } \lambda_k < \gamma^2; \\
    f_k t & \text{for } k \text{ such that } \lambda_k = \gamma^2; \\
    f_k \frac{\sin (\lambda_k - \gamma^2)^{1/2}t}{(\lambda_k - \gamma^2)^{1/2}} & \text{for } k \text{ such that } \lambda_k > \gamma^2.
\end{cases}
\]

The solution of (7.131) is then
\[
u(x, t) = \sum_{k=1}^{\infty} e^{-\gamma t}w_k(t)\phi_k(x).
\]
(7.137)

Inspection of the formula for $w_k$ shows that every coefficient of $\phi_k$ in (7.137) approaches 0 exponentially as $t \to \infty$. Disregarding factors independent of
time, the order of the damping increases until \( \lambda_k \) reaches \( \gamma^2 \); higher modes are all damped to the same extent. One point worth noting is that higher modes are damped in an oscillatory manner because of the cosine factor, whereas lower modes are damped monotonically. This oscillatory damping can be a serious structural drawback because of the fatigue stresses created. The difficulty can be remedied by using a mechanism of internal damping (see Exercise 7.29).

Returning to (7.137), it is clear that \( \lim_{t \to \infty} u(x, t) = 0 \) and that the decay is exponential. By virtue of (7.133), the same can be said for the solution of (7.132) and hence, by superposition, of the general initial value problem for the damped wave equation. The same conclusion holds for an unbounded region \( R \).

### 7.11 MONOCHROMATIC EXCITATION AND THE PRINCIPLE OF LIMITING ABSORPTION

We now turn to the inhomogeneous problem

\[
Lu = \frac{\partial^2 u}{\partial t^2} + 2\gamma \frac{\partial u}{\partial t} - \nabla^2 u = q(x, t), \quad x \text{ in } R, \quad t > 0;
\]

\[
u(x, t) = h(x, t), \quad x \text{ on } \sigma, \quad t > 0; \tag{7.138}
\]

\[
u(x, 0) = f_1(x), \quad \frac{\partial u}{\partial t}(x, 0) = f_2(x).
\]

In many physical situations both the forcing function \( q \) and the boundary schedule \( h \) are harmonic in time with the same frequency \( \omega \). We then say that the excitation is monochromatic. The case of general time dependence can be treated by writing \( q \) and \( h \) as Fourier integrals over frequency, analyzing each frequency separately, and then using superposition. We shall take the forcing function in the form

\[
q(x, t) = q(x)e^{-i\omega t}
\]

and \( h(x, t) \) in the form

\[
h(x, t) = h(x)e^{-i\omega t}.
\]

In actual physical problems with harmonic time dependence, the time factor is of the form \( \sin(\omega t + \psi) \), where \( \psi \) is a real phase angle; by taking a suitable linear combination of the real and imaginary parts of the solution corresponding to the time dependence \( e^{-i\omega t} \), we can obviously construct the solution for the time dependence, \( \sin(\omega t + \psi) \). For uniformity in the discussion the time factor will always be taken as \( e^{-i\omega t} \), \( \omega \) real, positive.

One can expect that, with \( q \) and \( h \) harmonic, the solution of (7.138) for large values of time will itself be harmonic with frequency \( \omega \). In any event, we can always write

\[
u(x, t) = U(x)e^{-i\omega t} + v(x, t), \tag{7.139}
\]
where $U$ and $v$ are to be determined from
\[ e^{-i\omega t}[ -\omega^2 U - 2i\gamma \omega U - \nabla^2 U ] + Lv = q(x)e^{-i\omega t}; \]
\[ v(x, t) + [U(x) - h(x)]e^{-i\omega t} = 0, \quad x \text{ on } \sigma, \quad t > 0; \]
\[ v(x, 0) = f_1(x) - U(x); \quad \frac{\partial v}{\partial t}(x, 0) = f_2(x) + i\omega U(x). \]

We are at liberty to choose $U$ and $v$ in a manner consistent with these conditions. Let us take $U$ to satisfy the space problem
\[ -\nabla^2 U - \lambda U = q(x), \quad x \text{ in } R; \quad \lambda = \omega^2 + 2i\gamma \omega; \]
\[ U(x) = h(x), \quad x \text{ on } \sigma, \] (7.140)
and then $v$ will satisfy
\[ Lv = \frac{\partial^2 v}{\partial t^2} + 2\gamma \frac{\partial v}{\partial t} - \nabla^2 v = 0, \quad x \text{ in } R, \quad t > 0; \]
\[ v(x, t) = 0, \quad x \text{ on } \sigma; \] (7.141)
\[ v(x, 0) = f_1(x) - U(x); \quad \frac{\partial v}{\partial t}(x, 0) = f_2(x) + i\omega U(x). \]

No matter what the initial values are, the solution of (7.141) tends to zero as $t \to \infty$, by our previous discussion of the homogeneous, damped wave equation. Further, it can be shown that (7.140) has one and only one solution, which we denote by $U(x, \lambda) = U(x, \omega^2 + 2i\gamma \omega)$. Therefore, for large values of $t$,
\[ u(x, t) \sim U(x, \lambda)e^{-i\omega t}. \] (7.142)

In many applications we are interested principally in the response long after the monochromatic excitation has begun to act. The steady-state solution is then given by (7.142), where $U$ is the solution of (7.140).

Our conclusion is based on the existence of a unique solution of (7.140). Let us scrutinize that assertion. If $R$ is a bounded region, $\lambda = \omega^2 + i\gamma \omega$ can not be an eigenvalue of (7.117), since all these eigenvalues are real and non-negative. Therefore, $\lambda$ is not in the spectrum and one can show that (7.140) has one and only one solution; in fact, using an expansion in the eigenfunctions of (7.117), we find explicitly
\[ U(x, \lambda) = \sum_{k=1}^{\infty} \frac{\varphi_k(x)}{\lambda_k - \lambda} \left[ \int_R q \varphi_k \, dx - \int_\sigma h \frac{\partial \varphi_k}{\partial n} \, dS \right], \quad \lambda = \omega^2 + 2i\gamma \omega. \] (7.143)

If $R$ is unbounded, but $q$ and $h$ vanish outside a finite portion of space, one can show that (7.140) has one and only one solution which vanishes at infinity. This result is certainly reasonable in view of the dissipation present. We shall have further remarks to make on this point in later sections.
In dealing with monochromatic excitation for the undamped wave equation, we must investigate (7.140) and (7.141) with \( \lambda = \omega^2 \), \( \omega \) real, positive. Strictly speaking, the solution of (7.141) no longer tends to 0 for large times, but if we endow the physical system with a small amount of dissipation, we may still regard (7.142) as the principal contribution to (7.139) for large \( t \). It remains then to discuss the solution of (7.140) when \( \lambda = \omega^2 \), \( \omega \) real, positive. If \( R \) is a bounded region and \( \lambda \) is not one of the positive eigenvalues of (7.117), we can still use (7.143) and \( U \) is unambiguously determined; if \( \lambda = \lambda_m \), (7.143) usually diverges (unless \( q \) and \( h \) are such that the bracketed term vanishes for \( k = m \)). This means that the asymptotic form of the solution of the time-dependent problem is no longer of the type (7.142) but instead increases with time, and after a short time the theory of small transverse deflections is no longer applicable—the membrane breaks or nonlinear effects take over. Of greater interest to us is the case of an unbounded region \( R \). The problem (7.117) no longer has any eigenvalues; instead, every nonnegative value of \( \lambda \) is in the continuous spectrum. Insofar as (7.140) is concerned, we can no longer expect a square integrable solution for arbitrary \( q \) and \( h \). What happens is that both the linearly independent solutions vanish at infinity but neither exhibits exponential decay at infinity. One must therefore find another criterion for choosing the physically relevant solution. Since every physical system has some damping in it, it is natural to regard the idealized, undamped case as the limit of the damped case as the dissipation coefficient \( \gamma \) approaches 0. This method for choosing a solution of (7.140) when \( \lambda \) is real, positive is known as the principle of limiting absorption. Thus we define \( U(x, \omega^2) \) by

\[
U(x, \omega^2) = \lim_{\gamma \to 0^+} U(x, \omega^2 + 2i\gamma \omega),
\]

which means that we must solve (7.140) when \( \lambda \) has a small positive imaginary part, \( \lambda = \omega^2 + i\alpha \), and then take the limit as \( \alpha \) tends to 0. Now for \( \varepsilon > 0 \), \( (\omega + i\varepsilon)^2 \) has a small positive imaginary part and \( (\omega + i\varepsilon)^2 \to \omega^2 \) as \( \varepsilon \to 0 \). Therefore, we also have

\[
U(x, \omega^2) = \lim_{\varepsilon \to 0^+} U[x, (\omega + i\varepsilon)^2],
\]

(7.144)

which is the form of the principle of limiting absorption that we shall employ. We shall see in later sections that one can characterize the desired physical solution of (7.140) for real, positive \( \lambda \) without appealing to limiting absorption.

**Energy Considerations**

Let \( u(x, t) \) be a real function satisfying the homogeneous, damped wave equation. If we multiply the differential equation by \( \partial u/\partial t \), we obtain, after some obvious simplifications,

\[
\frac{\partial}{\partial t} \varepsilon + \text{div} J = -2\gamma \left( \frac{\partial u}{\partial t} \right)^2,
\]

(7.145)
where

\[ \mathcal{E} = \frac{1}{2} \left( \frac{\partial u}{\partial t} \right)^2 + \frac{1}{2} \text{grad} \ u \cdot \text{grad} \ u \]

is the sum of the local kinetic and potential energy, and

\[ J = -\frac{\partial u}{\partial t} \text{grad} \ u \]  \hspace{1cm} (7.146)

is the local energy flux vector. In the expression for \( \mathcal{E} \), the significance of the terms depends on the physical problem at hand. For the vibrating membrane the first term is the kinetic energy and the second the potential energy; on the other hand, for problems in sound propagation, the roles are reversed, since \( u \) is the velocity potential, \( -\partial u/\partial t \) the pressure, and \( \text{grad} \ u \) the velocity.

Integrating (7.145) over a space region \( R \) containing no sources, we find

\[ \frac{dE}{dt} + \int_{\sigma} J \cdot n \ dS + 2\gamma \int_{R} \left( \frac{\partial u}{\partial t} \right)^2 \ dx = 0, \]  \hspace{1cm} (7.147)

where \( E \) is the total kinetic and potential energy within \( R \) at time \( t \), \( \sigma \) is the boundary of \( R \), and \( n \) is the outward normal to \( R \). Thus the rate of change of the total energy, the outward flux of energy through the boundary, and the rate of energy dissipation in \( R \) add up to 0. Therefore (7.147) describes conservation of energy for a source-free region \( R \).

Next we wish to calculate (7.146) in the monochromatic case. Then the real response \( u(x, t) \) can be written as either the real or imaginary part of \( U(x)e^{-i\omega t} \), where \( U(x) \) is a complex function satisfying (7.140). Of particular significance is the average \( J_a \) (over a period \( 2\pi/\omega \)) of the energy flux vector. An elementary calculation shows that

\[ J_a = \frac{\omega}{2} \text{Im} \left( -U \text{grad} \ U \right) = \frac{\omega}{4i} \left[ U \text{grad} \ U - U \text{grad} \ U \right]. \]  \hspace{1cm} (7.148)

If \( R \) is a region which is free of sources, we have

\[ \nabla^2 U + \lambda U = 0, \quad \nabla^2 U + \lambda U = 0, \]

and, hence

\[ \int_{\sigma} \left[ U \frac{\partial U}{\partial n} - U \frac{\partial U}{\partial n} \right] dS = (\lambda - \lambda) \int_{R} |U|^2 \ dx = -4i\gamma \omega \int_{R} |U|^2 \ dx. \]  \hspace{1cm} (7.149)

In particular for the undamped case,

\[ \int_{\sigma} J_a \cdot n \ dS = 0, \]  \hspace{1cm} (7.150)

if \( \sigma \) encloses no sources.

**EXERCISES**

7.19 Carry out the calculation leading to (7.148) and show that the same result is obtained whether we take \( u \) as the real or as the imaginary part of \( U(x)e^{-i\omega t} \).
7.20 By the energy method used at the beginning of Section 7.6, prove uniqueness for the damped wave equation (7.138) when \( R \) is a bounded region.

7.21 \textit{Kirchhoff's formula}. Consider a function \( u(x, t) \) satisfying the undamped wave equation

\[
\frac{\partial^2 u}{\partial t^2} - \nabla^2 u = q(x, t)
\]

in the whole of three-dimensional space for all \( t, -\infty < t < \infty \). Let \( R \) be a bounded space region with boundary \( \sigma \) and let \( D \) be a cylinder in space-time whose axis is in the time direction and whose bases are the region \( R \) at \( t = \tau \) and \( t = T \), where \( \tau \) is a large negative value of time. Apply Green's theorem (7.7) to the cylinder \( D \), with \( v = C_2(x_0, t_0 | x, t) \), \( \tau < t_0 < T \), to show that

\[
u(x_0, t_0) = \int_R \frac{q(x, t_0 - r)}{4\pi r} \, dx + \frac{1}{4\pi} \int_\sigma \left( \frac{1}{r} \frac{\partial u}{\partial n} \right) + \frac{1}{r^2} \frac{\partial r}{\partial n} [u] + \frac{1}{r} \frac{\partial r}{\partial n} \left( \frac{\partial u}{\partial t} \right) \right) dS_x, \tag{7.151}
\]

where \( r = |x - x_0| \) and the symbol \( [ \cdot ] \) stands for the quantity in brackets evaluated at position \( x \) and at the retarded time \( t_0 - r \).

7.22 Use Kirchhoff's formula (7.151) to obtain the solution (7.129) of the initial value problem.

7.23 By using Green's theorem (7.7) and the expression for

\[ C_2(x_0, y_0, t_0 | x, y, t) \]

from (7.130), obtain the solution of (7.109) when \( R \) is the whole of two-dimensional space.

7.24 Solve the general initial value problem for an infinite string with air resistance. Use the causal fundamental solution (5.169).

7.25 Obtain the causal fundamental solution for the equation of telegraphy (5.157) in the whole of 2-space and 1-space by the method of descent, starting from the causal solution for 3-space (5.168). Set \( \gamma = 1 \) in (5.157).

7.26 Derive the solution of (7.124) by three methods: (a) Laplace transform on time, (b) Fourier sine transform on space, and (c) reduction to a problem with a homogeneous boundary condition by setting

\[ u(x, t) = v(x, t) + h(t), \]

then applying (7.116) with \( g \) given by (7.123).
7.27 Find the causal Green's function for a semiinfinite string elastically supported at the end \( x = 0 \); that is,

\[
\frac{\partial u}{\partial x} (0, t) - \theta u(0, t) = 0
\]

is the boundary condition at \( x = 0 \). Consider only the undamped case.

7.28 Find the causal Green's function for the undamped wave equation for a finite string, \( 0 < x < l \), subject to the periodic boundary conditions

\[
u(0, t) = u(l, t), \quad \frac{\partial u}{\partial x} (0, t) = \frac{\partial u}{\partial x} (l, t).
\]

Use the method of images and also an expansion in the space eigenfunctions. By comparing the results, verify the Poisson summation formula (7.47).

7.29 *Internal damping.* Consider a system of \( n \) equal masses connected in a continuous chain by springs (with the same spring constant) and by dashpots (with equal damping). If the masses are constrained to move on a frictionless straight line, derive the equation of motion for the displacements of the masses from equilibrium. By an appropriate limiting process, pass to a continuous distribution of mass, and show that the displacement \( u(x, t) \) satisfies

\[
\frac{\partial^2 u}{\partial t^2} = a \frac{\partial^2 u}{\partial x^2} + 2b \frac{\partial^3 u}{\partial t \partial x^2},
\]

where \( a \) and \( b \) are known positive constants.

Consider the initial and boundary value problem for this equation (with \( a = 1 \)) in \( 0 < x < l, \ t > 0 \), with

\[
u(x, 0) = 0, \quad \frac{\partial u}{\partial t} (x, 0) = f(x), \quad u(0, t) = u(l, t) = 0.
\]

By separation of variables, find the solution. Show that all modes are damped but that higher modes are damped monotonically whereas lower ones are damped with oscillations.

7.30 Consider the following one-dimensional problem for the Helmholtz equation:

\[- \frac{d^2 U}{dx^2} - \lambda U = f_d(x), \quad \text{where} \ -\infty < x < \infty, \]

where \( \lambda \) is an arbitrary complex number and

\[
f_d(x) = \begin{cases} 
1, & |x| < a; \\
0, & |x| > a.
\end{cases}
\]

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Defining

\[ \sqrt{\lambda} = |\lambda|^{1/2} e^{i\theta}, \quad 0 \leq \theta < 2\pi, \]

show that the general solution of the differential equation is

\[ U = Ae^{i\sqrt{\lambda}x} + Be^{-i\sqrt{\lambda}x} + u_p, \]

where

\[ u_p(x) = \begin{cases} 
- \frac{1}{\lambda} + \frac{e^{i\sqrt{\lambda}a}}{\lambda} \cos \sqrt{\lambda}x, & |x| < a; \\
 i \sin \sqrt{\lambda}a (e^{i\sqrt{\lambda}|x|/\lambda}), & |x| > a. 
\end{cases} \]

Show that if \( \lambda \) is not in \([0, \infty)\), there is exactly one solution which vanishes at \(|x| = \infty\), and that this solution decays exponentially at infinity. Show that if \( \lambda \) is in \([0, \infty)\), there is no square-integrable solution unless \( a = n\pi/\sqrt{\lambda}, \ n = 1, 2, \ldots \). Use the principle of limiting absorption to pick out a unique solution when \( \lambda \) is in \([0, \infty)\).

7.12 GREEN'S FUNCTION FOR THE HELMHOLTZ OPERATOR AND APPLICATIONS

We have already seen that the Helmholtz operator \(-\nabla^2 - \lambda I\), where \( \lambda \) is a fixed complex number and \( I \) is the identity operator, plays a central role in the discussion of the wave equation and the diffusion equation. The principal cases are

1. \( \lambda \) real positive occurs in the study of monochromatic excitation for the undamped wave equation [see (7.140) with \( \gamma = 0 \)] and in the calculation of natural frequencies of vibrations of a bounded region (7.117).
2. \( \lambda = 0 \) is the problem of potential theory.
3. \( \lambda = \lambda_0 + ie, \ \lambda_0 > 0, \ e > 0 \), occurs in the study of monochromatic excitation for the damped wave equation (7.140).
4. \( \lambda = -k^2, \ k > 0 \), occurs for steady diffusion with absorption (or dissociation) proportional to the concentration. This case is also important in viscous flow.
5. \( \lambda = -s^2, \ \text{Re} \ s > 0 \), arises on taking the Laplace transform over time for the undamped wave equation.

Many of these cases obviously overlap and, except for cases 1 and 2, \( \lambda \) is not in \([0, \infty)\). For the kind of boundary conditions associated with the Helmholtz operator (\( u = 0 \) or \( \partial u/\partial n = 0 \), or \( \partial u/\partial n + \theta u = 0, \ \theta > 0 \)), the spectrum of \(-\nabla^2\), whether discrete or continuous, is confined to real non-negative \( \lambda \). In constructing the Green’s function for a bounded region, there is no difficulty as long as \( \lambda \) is not an eigenvalue (and since these eigenvalues are necessarily real and positive, the construction can always be performed for

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λ not real, positive). For an unbounded region there is a square-integrable Green’s function when λ is not in [0, ∞) but not when λ is in [0, ∞); for 0 < λ < ∞ we use the principle of limiting absorption to find the Green’s function. The case λ = 0 is best treated separately. We now proceed with some examples.

**Free-Space Fundamental Solution**

The free-space fundamental solution in n dimensions, \( E_n(x \mid x_0 ; \lambda) \), satisfies

\[
-\nabla^2 E_n - \lambda E_n = \delta(x - x_0), \quad x, x_0 \text{ in } R_n.
\]

Since the operator is invariant under translations and rotations, we may place the source at the origin and look for a solution which depends only on the radial coordinate. The calculations were performed in Section 5.8, but we restate the results.

**Case 1.** λ not in [0, ∞). We can then require \( E \) to behave well enough at infinity to be square integrable. We find,

(a) In three dimensions

\[
E_3(x \mid x_0 ; \lambda) = \frac{e^{i\sqrt{\lambda}|x-x_0|}}{4\pi|x-x_0|},
\]

where \( \sqrt{\lambda} \) is the unambiguous square root of λ having a positive imaginary part. This guarantees that \( E \) vanishes exponentially at \( \infty \) and is surely square integrable. The special case \( \lambda = -k^2 \), where \( k \) is positive, real, is covered by our formula, and we find

\[
E_3(x \mid x_0 ; -k^2) = \frac{e^{-k|x-x_0|}}{4\pi|x-x_0|}.
\]

(b) In two dimensions

\[
E_2(x \mid x_0 ; \lambda) = \frac{i}{4} H_0^{(1)}(\sqrt{\lambda}|x-x_0|),
\]

with the same understanding for \( \sqrt{\lambda} \). Again \( E \) will vanish exponentially at \( \infty \). For the special case \( \lambda = -k^2 \),

\[
E_2(x \mid x_0 ; -k^2) = \frac{i}{4} H_0^{(1)}(ik|x-x_0|) = \frac{1}{2\pi} K_0(k|x-x_0|),
\]

where \( K_0 \) is the Macdonald function.

(c) In one dimension

\[
E_1(x \mid x_0 ; \lambda) = \frac{ie^{i\sqrt{\lambda}|x-x_0|}}{2\sqrt{\lambda}},
\]
with $\sqrt{\lambda}$ defined above. For $\lambda = -k^2$,

$$E_1(x \mid x_0 ; -k^2) = \frac{e^{-k|x-x_0|}}{2k}.$$

**Case 2.** $\lambda$ in $(0, \infty)$; that is, $\lambda = \omega^2$, $\omega > 0$. By the principle of limiting absorption we have

$$E(x \mid x_0 ; \omega^2) = \lim_{\epsilon \to 0^+} E[x \mid x_0 ; (\omega + i\epsilon)^2].$$

From our definition of the square root,

$$\lim_{\epsilon \to 0^+} \sqrt{(\omega + i\epsilon)^2} = \omega,$$

so that formulas (7.152), (7.154), and (7.156) still apply. Thus

$$E_3(x \mid x_0 ; \omega^2) = \frac{e^{i\omega|x-x_0|}}{4\pi|x-x_0|},$$

(7.157)

which can be seen to represent an outgoing spherical wave if we adjoin the time factor $e^{-i\omega t}$.

Similarly,

$$E_2(x \mid x_0 ; \omega^2) = \frac{i}{4} H_0^{(1)}(\omega|x-x_0|),$$

(7.158)

and, by using the asymptotic form for the Hankel function of large argument,

$$E_2 \sim \frac{e^{i\pi/4}}{4} \left(\frac{2}{\pi \omega}\right)^{1/2} \frac{e^{i\omega|x-x_0|}}{|x-x_0|^{1/2}},$$

which represents an outgoing cylindrical wave. Also

$$E_1(x \mid x_0 ; \omega^2) = \frac{i}{2\omega} e^{i\omega|x-x_0|}$$

characterizes a plane wave traveling to the right for $x > x_0$ and to the left for $x < x_0$. As such, $E_1$ can also be considered to be an outgoing wave.

**Case 3.** $\lambda = 0$ (potential theory). The principle of limiting absorption does not apply in this case, since the physical motivation no longer exists ($\lambda = 0$ does not correspond to a dynamic problem). We can instead try to take the limit as $\lambda \to 0$ in our formulas for case 1. This works for $n \geq 3$ but not for $n = 1, 2$. In these last two cases the fundamental solution for potential theory is infinite at $|x| = \infty$ and cannot be directly obtained from (7.154) and (7.156). Instead we consider the gradient of $E$ (which corresponds to an electric field rather than a potential) and take the limit as $\lambda \to 0$ and then integrate to find
E. We then obtain the familiar results of potential theory

\[
E_3(x \mid x_0; 0) = \frac{1}{4\pi \vert x - x_0 \vert};
\]

\[
E_2(x \mid x_0; 0) = \frac{1}{2\pi} \log \frac{1}{\vert x - x_0 \vert};
\]

\[
E_1(x \mid x_0; 0) = -\frac{\vert x - x_0 \vert}{2}.
\]

**Addition Theorem for Cylindrical Waves**

Let \( r \) and \( \phi \) be the usual polar coordinates and let a source be placed at \( r = r_0, \phi = 0 \). From (7.154) we find the response

\[
E_2(r, \phi \mid r_0, 0; \lambda) = \frac{i}{4} H_0^{(1)}[\sqrt{\lambda(r^2 + r_0^2 - 2rr_0 \cos \phi)^{1/2}}].
\]  

(7.159)

For brevity, we sometimes denote the left side by \( E(r, \phi) \). We shall find it useful to obtain an alternative expression for \( E(r, \phi) \) as a series in the complete set of angular functions which arise on separation of the homogeneous Helmholtz equation in the polar coordinates \( r \) and \( \phi \). At first we suppose that \( \lambda \) is not in \([0, \infty)\). Then the homogeneous equation takes the form

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \phi^2} + \lambda u = 0, \quad -\pi < \phi < \pi, \quad 0 < r < \infty,
\]

and, writing \( u = R(r)\Phi(\phi) \), we find

\[
-\Phi'' = \mu\Phi,
\]

(7.160)

\[
-(rR')' - \lambda rR = \frac{\nu}{r} R,
\]

(7.161)

where \( \mu = -\nu \) is the separation constant that plays the role of an eigenvalue parameter (\( \lambda \) is fixed throughout and is not regarded as an eigenvalue parameter).

With (7.160) are associated the periodic boundary conditions

\[
\Phi(-\pi) = \Phi(\pi), \quad \Phi'(-\pi) = \Phi'(\pi),
\]

which determine the eigenvalues

\[
\mu = n^2, \quad n = 0, 1, 2, \ldots
\]

and the eigenfunctions \( e^{in\phi} \) and \( e^{-in\phi} \). It is more convenient to let \( n \) take on both negative and positive values and to regard \( e^{in\phi} \) as being the only eigenfunction attached to \( n^2 \). We then write

\[
E(r, \phi) = \sum_{n=-\infty}^{\infty} a_n(r) e^{in\phi},
\]
where
\[ a_n(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} E(r, \varphi) e^{-in\varphi} \, d\varphi. \]

The equation satisfied by \( E \) is
\[ -\frac{\partial}{\partial r} \left( r \frac{\partial E}{\partial r} \right) - \frac{1}{r} \frac{\partial^2 E}{\partial \varphi^2} - \lambda r E = \delta(r - r_0)\delta(\varphi), \]
\[ 0 < r, r_0 < \infty, \quad -\pi < \varphi < \pi. \quad (7.162) \]

Multiplying this equation by \((1/2\pi)e^{-in\varphi}\), integrating from \(-\pi\) to \(\pi\) in \(\varphi\), and using the periodicity of \(E\) and \(e^{-in\varphi}\), we obtain
\[-(ra_n')' - \lambda ra_n + \frac{n^2}{r} a_n = \frac{1}{2\pi} \delta(r - r_0), \quad 0 < r, r_0 < \infty.\]

The corresponding homogeneous equation is Bessel's equation of order \(n\) with parameter \(\lambda\). The solution which is finite at \(r = 0\) is \(J_n(\sqrt{\lambda} \, r)\), regardless of whether \(n\) is positive or negative. The only solution which vanishes exponentially at \(r = \infty\) is \(H_n^{(1)}(\sqrt{\lambda} \, r)\), where \(\sqrt{\lambda}\) is the square root with positive imaginary part. Thus after imposing continuity of \(a_n\) at \(r = r_0\), we have
\[ a_n = AJ_n(\sqrt{\lambda} \, r_<)H_n^{(1)}(\sqrt{\lambda} \, r_>). \]
The differential equation for \(a_n\) implies the jump condition
\[ a_n'(r_0^+) - a_n'(r_0^-) = -\frac{1}{2\pi r_0}, \]
so that
\[ A\sqrt{\lambda} \left[ J_n(\sqrt{\lambda} \, r_0)H_n^{(1)}(\sqrt{\lambda} \, r_0) - J_n'(\sqrt{\lambda} \, r_0)H_n^{(1)}(\sqrt{\lambda} \, r_0) \right] = -\frac{1}{2\pi r_0}. \quad (7.163) \]
By the Wronskian relationship (B.8) of Volume I, we find
\[ A = \frac{i}{4} \]
\[ E = \frac{i}{4} \sum_{n=-\infty}^{\infty} e^{in\varphi} J_n(\sqrt{\lambda} \, r_<)H_n^{(1)}(\sqrt{\lambda} \, r_>). \]
\[ J_{-n} = (-1)^n J_n, \quad H_{-n}^{(1)} = (-1)^n H_n^{(1)}, \quad (7.164) \]
we have
\[ E = \frac{i}{4} J_0(\sqrt{\lambda} \, r_<)H_0^{(1)}(\sqrt{\lambda} \, r_>) + \frac{i}{2} \sum_{n=1}^{\infty} \cos n\varphi \, J_n(\sqrt{\lambda} \, r_<)H_n^{(1)}(\sqrt{\lambda} \, r_>). \quad (7.165) \]
If the source is at \( \varphi = \varphi_0 \) instead of at \( \varphi = 0 \), we merely replace \( \varphi \) by \( \varphi - \varphi_0 \) in (7.164) and (7.165).

These expressions are apparently more complicated representations of (7.159), but we shall find a use for them in connection with problems dealing with the exterior of a cylinder (see Exercise 7.34). Some remarks are in order. If \( \lambda \) is positive and real, then (7.163) and (7.165) are still valid with \( \lambda \) the positive square root; this follows from the principle of limiting absorption. If \( \lambda \) is negative and real, \( \lambda = -k^2 \), we can express (7.163) in terms of modified Bessel functions as follows:

\[
E(r, \varphi | r_0, 0; -k^2) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{i\mu n} I_n(kr_0) K_n(kr). \tag{7.166}
\]

**Riemann Surface Fundamental Solution**

If the formula (7.159) for \( E \) is used to extend \( E \) for all \( \varphi \), we have a function which has period \( 2\pi \) in \( \varphi \). The physical plane corresponds to \( -\pi < \varphi < \pi \); other values of \( \varphi \) merely cover the physical plane again and again. As a function of \( \varphi \), \( -\infty < \varphi < \infty \), our extended \( E \) will have singularities at \( \varphi = 2n\pi \), where \( n \) is any integer. Thus the periodically extended function \( E \) satisfies

\[
-1 \frac{\partial}{r \partial r} \left( r \frac{\partial E}{\partial r} \right) - \frac{1}{r^2} \frac{\partial^2 E}{\partial \varphi^2} - \lambda E = \frac{\delta(r-r_0)}{r} \sum_{n=\infty}^{\infty} \delta(\varphi - 2n\pi),
0 < r, r_0 < \infty, -\infty < \varphi < \infty. \tag{7.167}
\]

We now ask whether we can find a fundamental solution whose only singularity in \( -\infty < \varphi < \infty \) is at \( \varphi = 0 \). This fundamental solution has no direct physical significance since it is not periodic in \( \varphi \), but we shall see later that it can help in the construction of physically relevant solutions of wedge problems. We shall denote the new fundamental solution (whose only singularity in \( -\infty < \varphi < \infty \) is at \( \varphi = 0 \)) by \( E_s \), and we shall call it the Riemann surface fundamental solution. The name stems from the complex variable analogy of a Riemann surface having an infinite number of sheets obtained by letting \( \varphi \) range from \( -\infty \) to \( \infty \): each \( 2\pi \) interval in \( \varphi \) of the form \( (2k - 1)\pi < \varphi < (2k + 1)\pi \) defining one sheet of the Riemann surface. The function \( E_s \) has a singularity on only one of these sheets (at \( \varphi = 0 \)), whereas \( E \) has a singularity on every sheet (at \( \varphi = 2k\pi \)). Thus \( E_s \) satisfies

\[
-1 \frac{\partial}{r \partial r} \left( r \frac{\partial E_s}{\partial r} \right) - \frac{1}{r^2} \frac{\partial^2 E_s}{\partial \varphi^2} - \lambda E_s = \frac{\delta(r-r_0)\delta(\varphi)}{r},
0 < r, r_0 < \infty, -\infty < \varphi < \infty. \tag{7.168}
\]

To calculate \( E_s \) we use an expansion in the angular functions arising from separation of the homogeneous equation in polar coordinates. The separation
again leads to (7.160) but now over the infinite interval \(-\infty < \varphi < \infty\), thereby suggesting the use of a Fourier transform. We define

\[
E^\wedge_s(r, \alpha) = \int_{-\infty}^{\infty} E_s(r, \varphi) e^{i\alpha \varphi} \, d\varphi,
\]

with the inversion

\[
E_s(r, \varphi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E^\wedge_s(r, \alpha) e^{-i\alpha \varphi} \, d\alpha.
\]

Multiply (7.168) by \(e^{i\alpha \varphi}\) and integrate from \(-\infty\) to \(\infty\). The integrated terms at \(\varphi = \pm \infty\) are set equal to 0 to obtain

\[
- \frac{d}{dr} \left( r \frac{dE^\wedge_s}{dr} \right) + \frac{\alpha^2}{r} E^\wedge_s - \lambda r E^\wedge_s = \delta(r - r_0), \quad 0 < r, r_0 < \infty.
\]

The solution of the homogeneous equation which is bounded at \(r = 0\) is \(J_{|a|}(\sqrt{\lambda} r)\), the absolute value sign being necessary since the Bessel function of negative index is infinite at \(r = 0\) unless the index is an integer. The solution with proper behavior at infinity can be written \(H^{(1)}_{|a|}(\sqrt{\lambda} r)\). Using the Wronskian relation (B.8) of Appendix B, Volume I, we find

\[
E^\wedge_s(r, \alpha) = \frac{i\pi}{2} J_{|a|}(\sqrt{\lambda} r_<) H^{(1)}_{|a|}(\sqrt{\lambda} r_>)
\]

and hence

\[
E_s(r, \varphi) = \frac{i}{4} \int_{-\infty}^{\infty} e^{-i\alpha \varphi} J_{|a|}(\sqrt{\lambda} r_<) H^{(1)}_{|a|}(\sqrt{\lambda} r_>) \, d\alpha, \quad (7.169)
\]
or

\[
E_s(r, \varphi) = \frac{i}{2} \int_{0}^{\infty} \cos \alpha \varphi \ J_s(\sqrt{\lambda} r_<) \, d\alpha. \quad (7.170)
\]

By the principle of limiting absorption, these formulas hold also for \(\lambda\) real, positive. Again, if the source is at \(\varphi = \varphi_0\), we replace \(\varphi\) by \(\varphi - \varphi_0\) throughout, so that

\[
E_s(r, \varphi | r_0, \varphi_0) = \frac{i}{4} \int_{-\infty}^{\infty} e^{-i\alpha(\varphi - \varphi_0)} J_{|a|}(\sqrt{\lambda} r_<) H^{(1)}_{|a|}(\sqrt{\lambda} r_>) \, d\alpha.
\]

From the definitions of \(E_s\) and \(E\), it is clear that

\[
E(r, \varphi | r_0, 0) = \sum_{n=-\infty}^{\infty} E_s(r, \varphi | r_0, 2n\pi)
\]

\[
= \sum_{n=-\infty}^{\infty} \frac{i}{4} \int_{-\infty}^{\infty} e^{i2\pi n} e^{-i\alpha \varphi} J_{|a|}(\sqrt{\lambda} r_<) H^{(1)}_{|a|}(\sqrt{\lambda} r_>) \, d\alpha.
\]
The last integral may be regarded as the Fourier transform of a function of \( \alpha \) evaluated when the transform variable is \( 2n\pi \). By the Poisson summation formula (7.47) we can then reduce (7.169) to

\[
E(r, \varphi | r_0, 0) = \frac{i}{4} \sum_{n=-\infty}^{\infty} e^{-in\varphi}J_{|n|}(\sqrt{\lambda}r_\varphi)H_{|n|}^{(1)}(\sqrt{\lambda}r_>)
\]

which, in view of (7.164), becomes

\[
E(r, \varphi | r_0, 0; \lambda) = \frac{i}{4} \sum_{n=-\infty}^{\infty} e^{in\varphi}J_n(\sqrt{\lambda}r_<)H_n^{(1)}(\sqrt{\lambda}r_>)
\]

in agreement with (7.163). We also note that when \( \lambda = -k^2 \), (7.169) becomes

\[
E_s(r, \varphi | r_0, 0; -k^2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\alpha \varphi}I_{|n|}(kr_<)K_{|n|}(kr_>)d\alpha. \tag{7.171}
\]

**Green's Function for a Wedge**

Consider the wedge \( 0 < \varphi < \psi \), \( r > 0 \), where \( \psi \) is the angle of the wedge. Thus \( \psi \leq 2\pi \), and the case \( \psi = 2\pi \) corresponds to the whole plane slit along a radial line. The Green's function corresponding to vanishing boundary conditions satisfies

\[
-\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial g}{\partial r} \right) - \frac{1}{r^2} \frac{\partial^2 g}{\partial \varphi^2} - \lambda g = \frac{\delta(r - r_0)\delta(\varphi - \varphi_0)}{r},
\]

\[
0 < r, r_0 < \infty, \quad 0 < \varphi, \varphi_0 < \psi. \tag{7.172}
\]

\[
g |_{\varphi=0} = 0; \quad g |_{\varphi=\psi} = 0.
\]

We shall construct \( g \) by images on the Riemann surface \( -\infty < \varphi < \infty \). If we place unit positive sources at \( \varphi_0 + 2m\psi \) and unit negative sources at \( -\varphi_0 + 2m\psi \), we have a complex of sources which is antisymmetric about \( \varphi = 0 \) and about \( \varphi = \psi \) and therefore the total field vanishes on \( \varphi = 0 \) and \( \varphi = \psi \). Moreover, there is a single source at \( \varphi = \varphi_0 \) in the physical region \( 0 < \varphi < \psi \). Thus

\[
g = \sum_{n=-\infty}^{\infty} [E_s(r, \varphi | r_0, \varphi_0 + 2m\psi) - E_s(r, \varphi | r_0, -\varphi_0 + 2m\psi)].
\]

By using the Poisson summation formula, we can easily transform this to

\[
g = \sum_{m=1}^{\infty} \frac{\pi i}{\psi} \sin \frac{m\pi \varphi_0}{\psi} \sin \frac{m\pi \varphi_0}{\psi} H_{m\pi/\psi}^{(1)}(\sqrt{\lambda}r_>)J_{m\pi/\psi}(\sqrt{\lambda}r_<), \tag{7.173}
\]

a formula still valid for \( \lambda \) real, positive.

Alternatively, this result can be obtained by expanding the solution of (7.172) in the \( \varphi \) eigenfunctions of the homogeneous equation which are
easily seen to be $\sin (m \pi \varphi /\psi)$. If $\lambda = - k^2$, we have
\[ g = \sum_{m=1}^{\infty} \frac{2}{\psi} \sin \frac{m \pi \varphi}{\psi} \sin \frac{m \pi \varphi_0}{\varphi} K_{m \pi /\psi}(kr_+)I_{m \pi /\psi}(kr_-). \quad (7.174) \]

Another Form for the Free-Space Green’s Function

We obtained (7.163) by expanding in the angular eigenfunctions found by separation of variables. It is natural to ask if one could not just as well start with the radial equation. The situation is now more complicated since the $\nu$ spectrum for (7.161) is continuous (remember that $\lambda$ is fixed and for simplicity we restrict ourselves to $\lambda = - k^2$, with $k$ real, positive). As was seen in Exercise 4.30, Volume I (where the parameters are labeled differently), the spectral decomposition leads to the Kantorovich-Lebedev transform pair
\[ \tilde{f}(\gamma) = \int_{0}^{\infty} \frac{f(r)}{r} K_{i\gamma}(kr) dr, \quad (7.175) \]
\[ f(r) = \frac{2}{\pi^2} \int_{0}^{\infty} \tilde{f}(\gamma)\gamma \sinh \pi \gamma K_{i\gamma}(kr) d\gamma, \quad (7.176) \]
where $K_{i\gamma}(kr)$ is Macdonald’s function of imaginary order, satisfying (7.161) with $\nu = \gamma^2$.

We shall apply this transform to (7.162), with $\lambda = - k^2$. Our procedure is entirely formal because some of the intermediate integrals diverge, but all is set right in the final answer, which can be shown to represent the solution. We multiply (7.162) by $K_{i\gamma}(kr)$ and integrate from 0 to $\infty$ in $r$ to obtain
\[ \int_{0}^{\infty} K_{i\gamma}(kr) \left[ - \frac{d}{dr} \left( r \frac{dE}{dr} \right) + k^2 r E \right] dr - \frac{d^2}{d\varphi^2} \tilde{E}(\gamma, \varphi) = K_{i\gamma}(kr_0) \delta(\varphi), \quad (7.177) \]
where $\tilde{E}$ is defined from
\[ \tilde{E}(\gamma, \varphi) = \int_{0}^{\infty} K_{i\gamma}(kr) \frac{E(r, \varphi)}{r} dr. \quad (7.178) \]

Integrating by parts, (7.177) becomes
\[ - \frac{d^2 \tilde{E}}{d\varphi^2} + \gamma^2 \tilde{E} = \delta(\varphi)K_{i\gamma}(kr_0), \quad -\pi < \varphi < \pi, \]
where we have set the integrated terms equal to zero without adequate justification. Taking into account the periodicity conditions
\[ \tilde{E}(\gamma, \pi) = \tilde{E}(\gamma, -\pi), \quad \frac{\partial \tilde{E}}{\partial \varphi} (\gamma, -\pi) = \frac{\partial \tilde{E}}{\partial \varphi} (\gamma, \pi), \]
we find
\[ \tilde{E}(\gamma, \varphi) = \frac{1}{2\gamma \sin \gamma \pi} \cosh \gamma(\pi - |\varphi|) K_{i\gamma}(kr_0), \quad -\pi < \varphi < \pi. \]
Hence, by the inversion formula (7.176),

\[ E(r, \varphi \mid r_0, 0) = \frac{1}{\pi^2} \int_0^\infty \cosh \gamma (\pi - |\varphi|) K_{i\gamma}(kr_0) K_{i\gamma}(kr) d\gamma, \]

or

\[ \frac{1}{2\pi} K_0[k(r^2 + r_0^2 - 2rr_0 \cos \varphi)^{1/2}] = \frac{1}{\pi^2} \int_0^\infty \cosh \gamma (\pi - |\varphi|) K_{i\gamma}(kr_0) K_{i\gamma}(kr) d\gamma, \quad -\pi < \varphi < \pi, \quad (7.179) \]

which is a known result and should be compared with the angular expansion (7.166).

If the source is located at \( \varphi = \varphi_0 \), we have

\[ \frac{1}{2\pi} K_0[k[r^2 + r_0^2 - 2rr_0 \cos(\varphi - \varphi_0)]^{1/2}] = \frac{1}{\pi^2} \int_0^\infty \cosh \gamma (\pi - |\varphi - \varphi_0|) K_{i\gamma}(kr_0) K_{i\gamma}(kr) d\gamma, \quad (7.180) \]

where the angle \( \varphi \) is measured so that \( -\pi < \varphi - \varphi_0 < \pi \).

The expansion is useful in calculating the Green's function for a source in front of a knifedge. Using the polar coordinates \( 0 < \varphi < 2\pi \), the knifedge is the half-line \( \varphi = 0 \) or \( \varphi = 2\pi \). The source is located at \( (r_0, \pi) \) as in Figure 7.9. The boundary condition is that the total field vanishes on the knifedge. We write the total field \( g \) as the sum of the free-space fundamental solution \( E \) and of a scattered field \( v \),

\[ g(r, \varphi \mid r_0, \pi) = E(r, \varphi \mid r_0, \pi) + v(r, \varphi, r_0). \quad (7.181) \]

Now \( v \) satisfies the homogeneous Helmholtz equation for \( 0 < \varphi < 2\pi \), and takes on known values at \( \varphi = 0 \) and \( \varphi = 2\pi \). Thus

![Figure 7.9](www.MathSchoolInternational.com)
\[ -\frac{\partial}{\partial r} \left( r \frac{\partial v}{\partial r} \right) - \frac{1}{r} \frac{\partial^2 v}{\partial \varphi^2} + k^2 rv = 0, \quad v|_{\varphi=0} = -E(r,0|r_0,\pi), \]
\[ v|_{\varphi=2\pi} = -E(r,2\pi|r_0,\pi). \] (7.182)

Taking the Kantorovich-Lebedev transform, we obtain
\[ -\frac{d^2 \tilde{v}}{d\varphi^2} + \gamma^2 \tilde{v} = 0, \]
or
\[ \tilde{v} = A e^{-\gamma \varphi} + B e^{\gamma \varphi}. \]

At \( \varphi = 0 \) we have
\[ A + B = -\tilde{E}(\gamma,0|r_0,\pi) = -\frac{1}{2\gamma \sin \gamma \pi} K_{i\gamma}(kr_0). \]

At \( \varphi = 2\pi \),
\[ Ae^{-2\gamma \pi} + Be^{2\gamma \pi} = -\tilde{E}(\gamma,2\pi|r_0,\pi) = -\frac{1}{2\gamma \sin \gamma \pi} K_{i\gamma}(kr_0). \]

We deduce that
\[ \tilde{v} = \frac{1}{\pi^2} \frac{\sinh \gamma(\varphi - 2\pi) - \sinh \gamma \varphi}{\sinh 2\gamma \pi} K_{i\gamma}(kr_0), \]
from which \( v \) can be calculated by (7.176) and then \( g \) from (7.181). By contour integration the result can be transformed to
\[ g = \sum_{m=1}^{\infty} \frac{1}{\pi} \sin \frac{m\varphi}{2} \sin \frac{m\pi}{2} K_{m/2}(kr_>) I_{m/2}(kr_<), \] (7.183)
in agreement with (7.174), with \( \psi = 2\pi \).

**Hankel Transform**

Consider a three-dimensional problem for the operator \(-\nabla^2 - \lambda I\), when the solution is known from additional symmetry conditions to depend only on the cylindrical coordinates \( r \) and \( z \). This is the case for all axisymmetric problems, with \( z \) the axis of symmetry. Separating the homogeneous equation in this coordinate system, we obtain the following ordinary differential equations
\[ -Z'' - \mu Z = 0, \] (7.184)
\[ -(rR)' - \lambda r R - vr R = 0, \] (7.185)
where \( \mu = -\nu \) is the separation constant playing the role of an eigenvalue parameter (\( \lambda \) is fixed throughout).
To solve the original partial differential equation we may expand in either the z or r direction. If the problem is to be solved for the r domain, $0 < r < \infty$, then (7.185) leads to the Hankel transform of order zero (see Exercise 4.26). By definition the Hankel transform of order zero of $f(r)$ is

$$F_H(\gamma) = \int_0^\infty r J_0(\gamma r) f(r) dr,$$  \hspace{1cm} (7.186)

and we recover $f$ from $F_H$ by the inversion formula

$$f(r) = \int_0^\infty \gamma J_0(\gamma r) F_H(\gamma) d\gamma.$$  \hspace{1cm} (7.186a)

**Source Ring in Free Space**

With these preparations we are now ready to attack a specific problem. Suppose we have a ring of sources of total strength 1 uniformly distributed on $r = a$, $z = 0$. Then the part of the ring lying within the angle $d\varphi_0$ at $\varphi_0$ carries a concentrated source of strength $d\varphi_0/2\pi$ and its volume density is therefore

$$\frac{1}{2\pi} d\varphi_0 \frac{\delta(r - a)\delta(z)\delta(\varphi - \varphi_0)}{r}.$$  

By integrating from $\varphi_0 = -\pi$ to $\varphi_0 = \pi$, we find the volume source density of the ring to be

$$\frac{\delta(r - a)\delta(z)}{2\pi r}.$$  

The response $u$ due to the ring of sources satisfies

$$-\nabla^2 u - \lambda u = \frac{\delta(r - a)\delta(z)}{2\pi r}, \hspace{1cm} -\infty < z < \infty, \hspace{0.2cm} 0 < r < \infty,$$  \hspace{1cm} (7.187)

where at first $\lambda$ is not in [0, $\infty$). The problem is clearly axisymmetric and the solution $u$ will be independent of the polar angle $\varphi$.

**Hankel Transform of Order Zero on the Radial Coordinate**

In cylindrical coordinates, (7.187) becomes

$$-\frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) - r \frac{\partial^2 u}{\partial z^2} - \lambda ru = \frac{\delta(r - a)\delta(z)}{2\pi}.$$  \hspace{1cm} (7.188)

Multiply both sides by $J_0(\gamma r)$ and integrate from $r = 0$ to $r = \infty$ to obtain

$$-\int_0^\infty J_0(\gamma r) \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) dr - \frac{d^2}{dz^2} U_H - \lambda U_H = \frac{J_0(\gamma a)\delta(z)}{2\pi},$$  \hspace{1cm} (7.189)
where

$$U_H(\gamma, z) = \int_0^\infty r J_0(\gamma r) u(r, z) dr$$

is the Hankel transform of order 0 of $u(r, z)$, with $z$ a parameter. Integrating by parts in (7.189) and discarding the integrated terms, we find

$$-\int_0^\infty u \frac{d}{dr} \left( r \frac{d J_0(\gamma r)}{dr} \right) dr - \frac{d^2 U_H}{dz^2} - \lambda U_H = \frac{J_0(\gamma a) \delta(z)}{2\pi}.$$  

Now $R(r) = J_0(\gamma r)$ satisfies (7.185), with $\lambda + \nu = \gamma^2$. Thus

$$-\frac{d^2 U_H}{dz^2} + (\gamma^2 - \lambda) U_H = \frac{J_0(\gamma a) \delta(z)}{2\pi}. \quad (7.190)$$

Letting $\sqrt{\gamma^2 - \lambda}$ be the square root with positive real part (which is always possible, since $\gamma^2$ is real positive and $\lambda$ is not), and, requiring that $U_H$ vanish at $|z| = \infty$, we have

$$U_H(\gamma, z) = \frac{e^{-\sqrt{\gamma^2 - \lambda} |z|}}{\sqrt{\gamma^2 - \lambda}} \frac{J_0(\gamma a)}{4\pi}. \quad (7.191)$$

By the inversion formula (7.186a), we find

$$u(r, z) = \frac{1}{4\pi} \int_0^\infty \gamma J_0(\gamma r) J_0(\gamma a) \frac{e^{-\sqrt{\gamma^2 - \lambda} |z|}}{\sqrt{\gamma^2 - \lambda}} d\gamma. \quad (7.192)$$

If $a \to 0$, the ring degenerates into a unit source at the origin and $u(r, z)$ must reduce to (7.152). This yields the interesting relation (due to Sommerfeld)

$$\frac{e^{\sqrt{\lambda \gamma^2 + z^2}}}{\sqrt{r^2 + z^2}} = \int_0^\infty \gamma J_0(\gamma r) \frac{e^{-\sqrt{\gamma^2 - \lambda} |z|}}{\sqrt{\gamma^2 - \lambda}} d\gamma. \quad (7.193)$$

We remind the reader that this is valid for $\lambda$ not in $[0, \infty)$ and with $\sqrt{\lambda}$ chosen with positive imaginary part and $\sqrt{\gamma^2 - \lambda}$ chosen with positive real part. Two particular cases should be mentioned. If $\lambda = -k^2$, $k$ real, positive, we find

$$\frac{e^{-k \sqrt{\gamma^2 + z^2}}}{\sqrt{r^2 + z^2}} = \int_0^\infty \gamma J_0(\gamma r) \frac{e^{-\sqrt{\gamma^2 + k^2} |z|}}{\sqrt{\gamma^2 + k^2}} d\gamma, \quad (7.194)$$

where $\sqrt{\gamma^2 + k^2}$ is the positive square root of the positive number $\gamma^2 + k^2$.

On the other hand, the result for $\lambda$ real and positive is obtained by the principle of limiting absorption. We need to calculate

$$\lim_{\epsilon \to 0^+} \sqrt{\gamma^2 - (\lambda + i\epsilon)},$$

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where the square root has positive real part. Thus, for \( \lambda < \gamma^2 \), we have
\[
\lim_{\varepsilon \to 0^+} \sqrt{\gamma^2 - (\lambda + i\varepsilon)} = (\gamma^2 - \lambda)^{1/2},
\]
and, for \( \lambda > \gamma^2 \),
\[
\lim_{\varepsilon \to 0^+} \sqrt{\gamma^2 - (\lambda + i\varepsilon)} = -i(\lambda - \gamma^2)^{1/2}.
\]
Therefore, for \( \lambda \) real positive, (7.192) becomes
\[
e^{\sqrt{\lambda^2 + z^2}} \frac{1}{\sqrt{\lambda^2 + z^2}} = i \int_0^{\frac{\lambda}{\gamma^2}} \gamma J_0(\gamma r) \frac{\gamma (\lambda - \gamma^2)^{1/2} |z|}{(\lambda - \gamma^2)^{1/2}} d\gamma + \int_{\frac{\lambda}{\gamma^2}}^{\infty} \gamma J_0(\gamma r) \frac{\gamma (\lambda - \gamma^2)^{1/2} |z|}{(\lambda - \gamma^2)^{1/2}} d\gamma,
\]
(7.194)
where all square roots that appear are positive square roots of positive numbers.

The formula (7.193) can be shown to be valid even for \( k = 0 \), when it reduces to
\[
\frac{1}{\sqrt{r^2 + z^2}} = \int_0^\infty J_0(\gamma r) e^{-\gamma |z|} d\gamma.
\]
(7.195)

**Fourier Transform on z**

The solution of (7.188) can also be obtained in a different form by expanding in the \( z \) direction. Since the interval in \( z \) is from \(-\infty\) to \( \infty \), we are led by (7.184) to the use of a Fourier transform. Multiply (7.188) by \( e^{iaz} \) and integrate from \(-\infty\) to \( \infty \) in \( z \) to obtain
\[
-\frac{d}{dr} \left( r \frac{du^\wedge}{dr} \right) + (\alpha^2 - \lambda)ru^\wedge = \frac{\delta(r - a)}{2\pi}, \quad 0 < r < \infty,
\]
where
\[
u^\wedge(r, \alpha) = \int_{-\infty}^{\infty} e^{iaz} u(r, z) dz.
\]
Letting \( \sqrt{\alpha^2 - \lambda} \) be the square root with positive real part, and requiring that \( u^\wedge \) vanish at \( r = \infty \), we find
\[
u^\wedge(r, \alpha) = \frac{1}{2\pi} I_0(\sqrt{\alpha^2 - \lambda} r_\wedge) K_0(\sqrt{\alpha^2 - \lambda} r_\wedge),
\]
where \( r_\wedge = \min(r, a) \), \( r_\wedge = \max(r, a) \). Hence
\[
u(r, z) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} e^{-iaz} I_0(\sqrt{\alpha^2 - \lambda} r_\wedge) K_0(\sqrt{\alpha^2 - \lambda} r_\wedge) da.
\]
(7.196)
As $a \to 0$, we have again the case of a unit source at the origin, so that

$$\frac{e^{i\sqrt{\lambda} r^2 + z^2}}{\sqrt{r^2 + z^2}} = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-iaz} K_0(\sqrt{\alpha^2 - \lambda} r)\,d\alpha. \tag{7.197}$$

If $\lambda = -k^2$, $k$ real, positive,

$$\frac{e^{-k\sqrt{r^2 + z^2}}}{\sqrt{r^2 + z^2}} = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-iaz} K_0(\sqrt{\alpha^2 + k^2} r)\,d\alpha, \tag{7.198}$$

where $\sqrt{\alpha^2 + k^2}$ is the positive square root of $\alpha^2 + k^2$. If $\lambda$ is real, positive, we find, by the principle of limiting absorption,

$$\frac{e^{i\sqrt{\lambda} r^2 + z^2}}{\sqrt{r^2 + z^2}} = \frac{2}{\pi} \int_0^{\lambda^{1/2}} \cos az \, K_0[-i(\lambda - \alpha^2)^{1/2} r]\,d\alpha$$

$$+ \frac{2}{\pi} \int_{\lambda^{1/2}}^{\infty} \cos az \, K_0[(\alpha^2 - \lambda)^{1/2} r]\,d\alpha. \tag{7.199}$$

We also point out that the solution (7.196) can be further expanded by using a Hankel transform, just as (7.191) can be further expanded in a Fourier integral on $z$. This leads to the full bilinear expansion for the solution of (7.188),

$$u(r, z) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} d\alpha \int_0^{\infty} d\gamma \ \frac{\gamma J_0(\gamma r) J_0(\gamma a)}{\alpha^2 + \gamma^2 - \lambda} \ e^{-iaz}. \tag{7.200}$$

A further consequence of (7.197) is that

$$K_0(\sqrt{\alpha^2 - \lambda} r) = \frac{1}{2} \int_{-\infty}^{\infty} e^{iaz} \frac{e^{i\sqrt{\alpha^2 - z^2}}}{\sqrt{r^2 + z^2}} \,dz,$$

and, with $a = 0$,

$$K_0(\sqrt{-\lambda} r) = \frac{1}{2} \int_{-\infty}^{\infty} \frac{e^{i\sqrt{\lambda} r^2 + z^2}}{\sqrt{r^2 + z^2}} \,dz.$$

Setting $\lambda = -k^2$, $k > 0$, we obtain the well-known integral representation of the Macdonald function,

$$K_0(kr) = \frac{1}{2} \int_{-\infty}^{\infty} \frac{e^{-kr^2 + z^2}}{\sqrt{r^2 + z^2}} \,dz. \tag{7.201}$$

By the principle of limiting absorption, we also find for $\lambda$ real, positive,

$$H_0^{(1)}(\sqrt{\lambda} r) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{e^{i\sqrt{\lambda} r^2 + z^2}}{\sqrt{r^2 + z^2}} \,dz. \tag{7.202}$$

where $\sqrt{\lambda}$ is the positive square root.
Consider next the problem of a ring of sources for the heat equation. The temperature \( v(r, z, t) \) satisfies

\[
\frac{\partial v}{\partial t} - \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v}{\partial r} \right) - \frac{\partial^2 v}{\partial z^2} = 0, \quad t > 0, \quad -\infty < z < \infty, \quad r > 0,
\]

(7.203)

\[
v(r, z, 0) = \frac{\delta(r - a)\delta(z)}{2\pi r}.
\]

First we find the solution by breaking up the ring into point sources. The temperature at time \( t \) at \((r, 0, z)\) is then given by

\[
\frac{1}{2\pi} \int_0^{2\pi} e^{-R^2/4t} \frac{e^{-R^2/4t}}{(4\pi t)^{3/2}} \, d\varphi_0,
\]

where \( d\varphi_0 \) is the element of angle subtending an infinitesimal arc of the ring at \((a, \varphi_0, 0)\), and \( R \) is the distance between \((r, 0, z)\) and \((a, \varphi_0, 0)\); that is,

\[
R^2 = z^2 + r^2 + a^2 - 2ar \cos \varphi_0.
\]

Therefore, since the temperature \( v \) is independent of \( \varphi \), we have

\[
v(r, z, t) = \frac{1}{2\pi} \frac{1}{(4\pi t)^{3/2}} \int_0^{2\pi} e^{-(z^2 + r^2 + a^2)/4t} e^{(ar/2t)\cos \varphi_0} \, d\varphi_0
\]

\[
= \frac{1}{(4\pi t)^{3/2}} e^{-(z^2 + r^2 + a^2)/4t} e^{r \frac{a}{2t}}.
\]

(7.204)

We observe that this reduces to the correct expression for a point source as \( a \to 0 \).

One can write the solution in a different form by applying a Laplace transform to (7.203). Then, with

\[
\bar{v}(r, z, s) = \int_0^\infty v(r, z, t) e^{-st} \, dt,
\]

we find

\[
-\nabla^2 \bar{v} + s\bar{v} = \frac{\delta(r - a)\delta(z)}{2\pi r},
\]

whose solution, by (7.191), is

\[
\bar{v}(r, z, s) = \frac{1}{4\pi} \int_0^\infty \gamma J_0(\gamma r)J_0(\gamma a) \frac{e^{-\sqrt{s} + s|z|}}{\sqrt{\gamma^2 + s}} \, d\gamma,
\]

which is valid for \( s \) not in \((-\infty, 0]\), with \( \sqrt{\gamma^2 + s} \) the square root with positive real part. The Laplace inversion requires only a knowledge of the inverse of

\[
\frac{e^{-\sqrt{s} + s|z|}}{\sqrt{\gamma^2 + s}},
\]

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which, by elementary considerations, is
\[ \frac{2e^{-\gamma^2/4t}}{\sqrt{4\pi t}}, \quad t > 0. \]

Therefore,
\[ u(r, z, t) = \frac{2e^{-\gamma^2/4t}}{(4\pi)^{3/2} t^{1/2}} \int_0^\infty \gamma J_0(\gamma r) J_0(\gamma a) e^{-\gamma^2 t} \, d\gamma. \]  (7.205)

7.13 HALF-PLANE EXCITED BY A LINE SOURCE OR A PLANE WAVE

The problem of this section has already been treated in a preliminary way in previous sections [see, for instance, (7.174) with \( \psi = 2\pi \) and (7.182) and (7.183)]. We now go into more detail and express the solution in more tractable form. Let the half-plane be described in Cartesian coordinates by \( y = 0, \ x > 0, \ -\infty < z < \infty \), so that its trace in a plane normal to the \( z \) axis is the positive \( x \) axis. Using polar coordinates \( (r, \varphi) \), the trace of the half-plane corresponds to \( \varphi = 0 \) and \( \varphi = 2\pi \) and the trace of the line source is the point \( (r_0, \varphi_0) \), as in Figure 7.10.

The boundary value problem corresponding to a zero boundary condition on the half-plane is
\[ -1 \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial g}{\partial r} \right) - \frac{1}{r^2} \frac{\partial^2 g}{\partial \varphi^2} - \lambda g = \frac{\delta(r - r_0) \delta(\varphi - \varphi_0)}{r}, \quad 0 \leq \varphi, \varphi_0 < 2\pi, \quad 0 < r, r_0 < \infty; \quad (7.206) \]
\[ g(r, 0 | r_0, \varphi_0) = g(r, 2\pi | r_0, \varphi_0) = 0. \]

---

**FIGURE 7.10**
Separating the homogeneous equation in polar coordinates, we find for the angular part

$$-\Phi'' - \mu \Phi = 0, \quad 0 < \varphi < 2\pi; \quad \Phi(0) = \Phi(2\pi) = 0.$$

The eigenvalues are $\mu_n = n^2/4$, $n = 1, 2, \ldots$, and the eigenfunctions are

$$\Phi_n(\varphi) = \sin \frac{n\varphi}{2}, \quad n = 1, 2, \ldots.$$

This suggests expanding the solution of (7.206) in the complete set $\sin (n\varphi/2)$; thus we write

$$g = \sum_{n=1}^{\infty} g_n \sin \frac{n\varphi}{2}, \quad g_n = \frac{1}{\pi} \int_0^{2\pi} g \sin \frac{n\varphi}{2} \, d\varphi.$$

Multiplying the differential equation by $(1/\pi) \sin (n\varphi/2)$, and integrating from $\varphi = 0$ to $\varphi = 2\pi$, we obtain

$$-\frac{1}{r} \frac{d}{dr} \left( r \frac{dg_n}{dr} \right) + \frac{n^2}{4r^2} g_n - \lambda g_n = \frac{\delta(r - r_0)}{\pi r} \sin \frac{n\varphi_0}{2}.$$

Since $g_n$ is bounded at $r = 0$ and must vanish at $r = \infty$, we have

$$g_n = \begin{cases} AJ_{n/2}(\sqrt{\lambda} r), & r < r_0, \\ BH_{n/2}^{(1)}(\sqrt{\lambda} r), & r > r_0, \end{cases}$$

where $\sqrt{\lambda}$ is the square root with positive imaginary part. By the usual arguments we find

$$g_n = \frac{i}{2} J_{n/2}(\sqrt{\lambda} r_<) H_n^{(1)}(\sqrt{\lambda} r_>) \sin \frac{n\varphi_0}{2},$$

where $r_\leq = \min (r, r_0)$ and $r_\geq = \max (r, r_0)$, and, therefore,

$$g = \sum_{n=1}^{\infty} \frac{i}{2} \sin \frac{n\varphi}{2} \sin \frac{n\varphi_0}{2} J_{n/2}(\sqrt{\lambda} r_<) H_n^{(1)}(\sqrt{\lambda} r_>, \quad \text{(7.207)}$$

which agrees with (7.173) with $\psi = 2\pi$.

The last expression can be split up into parts with odd and even symmetry, respectively, about the $x$ axis, that is,

$$g = g_1 + g_2,$$

where

$$g_1 = \frac{i}{2} \sum_{n=1}^{\infty} \sin n\varphi \sin n\varphi_0 J_{n}(\sqrt{\lambda} r_<) H_n^{(1)}(\sqrt{\lambda} r_>) \quad \text{(7.208)}$$

$$g_2 = \frac{i}{2} \sum_{n=0}^{\infty} \sin (n + \frac{1}{2})\varphi \sin (n + \frac{1}{2})\varphi_0 J_{n+(1/2)}(\sqrt{\lambda} r_<) H_n^{(1)}(\sqrt{\lambda} r_>) \quad \text{(7.209)}$$
The physical interpretation of \( g_1 \) and \( g_2 \) is straightforward: \( g_1 \) corresponds to an excitation consisting of a positive source (of strength \( \frac{1}{2} \)) at \((r_0, \varphi_0)\) and of a negative source of like strength at the image point \((r_0, 2\pi - \varphi_0)\), whereas \( g_2 \) corresponds to an excitation due to a positive source of strength \( \frac{1}{2} \) at \((r_0, \varphi_0)\) and a like positive source at \((r_0, 2\pi - \varphi_0)\).

The antisymmetry of \( g_1 \) about the \( x \) axis implies that it must vanish on the whole \( x \) axis and therefore by the method of images

\[
g_1 = \frac{i}{8} [H_0^{(1)}(\sqrt{\lambda} R) - H_0^{(1)}(\sqrt{\lambda} R^*)],
\]

where (see Figure 7.10) \( R \) is the distance between \((r, \varphi)\) and \((r_0, \varphi_0)\), and \( R^* \) is the distance between \((r, \varphi)\) and the image source \((r_0, 2\pi - \varphi_0)\). The function \( g_2 \) is even about the \( x \) axis and since it is continuously differentiable on the negative \( x \) axis, we must have \( \partial g_2 / \partial y = 0, \ x < 0, \ y = 0 \). Thus \( g_2 \) satisfies the following boundary value problem for the region \( 0 < \varphi < \pi \),

\[
- \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial g_2}{\partial r} \right) - \frac{1}{r^2} \frac{\partial^2 g_2}{\partial \varphi^2} - \lambda g = \frac{\delta(r - r_0)\delta(\varphi - \varphi_0)}{2r},
\]

\( 0 < \varphi, \varphi_0 < \pi, \ 0 < r, r_0 < \infty; \)

\[
g_2 = 0, \ \varphi = 0; \ \frac{\partial g_2}{\partial \varphi} = 0, \ \varphi = \pi.
\]

Leaving aside these considerations, we return to the explicit form (7.209) for \( g_2 \), which can be rewritten

\[
g_2 = s(r, r_0, \varphi - \varphi_0) - s(r, r_0, \varphi + \varphi_0),
\]

where

\[
s(r, r_0, \psi) = \frac{i}{4} \sum_{n=0}^{\infty} \cos (n + \frac{1}{2}) \psi J_{n+(1/2)}(\sqrt{\lambda} r_<) H_{n+(1/2)}^{(1)}(\sqrt{\lambda} r_>).
\]

To find a more suitable form for \( g_2 \), we will have to transform the expression for \( s \). We shall use a technique due to H. Levine. From Exercise 7.31, we find the following integral representation of the product of cylinder functions,

\[
J_{n+(1/2)}(\sqrt{\lambda} r_<) H_{n+(1/2)}^{(1)}(\sqrt{\lambda} r_>) = \frac{\sqrt{rr_0}}{\pi i} \int_0^\pi \frac{e^{i\sqrt{\lambda} \rho}}{\rho} \sin \theta \ P_n(\cos \theta) d\theta,
\]

where \( P_n \) is the Legendre polynomial of degree \( n \) and

\[
\rho^2 = r^2 + r_0^2 - 2rr_0 \cos \theta.
\]

Substituting this integral representation in (7.213), we are left with the problem of explicitly summing the series

\[
T(\psi, \theta) = \sum_{n=0}^{\infty} \cos (n + \frac{1}{2}) \psi \ P_n(\cos \theta).
\]

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From the generating function for Legendre polynomials [see Appendix A, equation (A.12)], one can derive Mehler’s integral representation

\[ P_n(\cos \theta) = \frac{\sqrt{2}}{\pi} \int_0^\theta \frac{\cos(n + \frac{1}{2})\xi}{\sqrt{\cos \xi - \cos \theta}} d\xi, \quad 0 < \theta < \pi, \]

which, in turn, leads to

\[ \sum_{n=0}^\infty \cos(n + \frac{1}{2})\psi P_n(\cos \theta) = \begin{cases} \frac{1}{2(\cos \psi - \cos \theta)}^{1/2}, & 0 \leq \psi < \theta < \pi; \\ 0, & 0 < \theta < \psi \leq \pi. \end{cases} \]

Here use has been made of the relation

\[ \sum_{n=0}^\infty \cos(n + \frac{1}{2})\psi = \frac{2}{\pi} \sum_{n=0}^\infty \cos(n + \frac{1}{2})\xi \delta(\psi - \xi), \quad 0 < \psi, \xi < \pi, \]

which is a direct consequence of the fact that \(\{(2/\pi)^{1/2} \cos(n + \frac{1}{2})\psi\}\) are the normalized eigenfunctions of the Sturm-Liouville problem, \(-u'' - \lambda u = 0, u'(0) = u(\pi) = 0\). Therefore,

\[ s(r, r_0, \psi) = \frac{\sqrt{rr_0}}{4\pi} \int_\psi^\pi R \left( \sin \theta \right) \frac{e^{i\sqrt{\lambda}R}}{[2(\cos \psi - \cos \theta)]^{-1/2}} d\theta, \quad 0 < \psi < \pi. \]

We now make the change of variables,

\[ \frac{v^2}{2rr_0} = \cos \psi - \cos \theta, \]

to obtain

\[ s(r, r_0, \psi) = \frac{1}{4\pi} \int_0^{2\sqrt{rr_0} \cos(\psi/2)} e^{i\sqrt{\lambda}(p^2 + v^2)^{1/2}} \frac{[p^2 + v^2]^{1/2}}{[p^2 + v^2]^{1/2}} dv, \quad (7.215) \]

where

\[ p^2 = r^2 + r_0^2 - 2rr_0 \cos \psi. \]

We can easily extend the range of validity of (7.215), which at first only holds for \(0 \leq \psi \leq \pi\). Formula (7.213) defines an even function of \(\psi\), as does (7.215), so that the latter is also correct for \(-\pi \leq \psi \leq 0\). Moreover, we see from (7.213) that \(s(r, r_0, \psi + 2\pi) = -s(r, r_0, \psi)\), a property which is also true for (7.215), thereby extending its validity to \(-2\pi \leq \psi \leq 2\pi\). Since both (7.213) and (7.215) have period \(4\pi\), the two expressions are equal for all \(\psi\).

From (7.210), (7.212), and (7.215), we obtain

\[ g(r, \varphi | r_0, \varphi_0) = \frac{i}{8} \left[ H_0^{(1)}(\sqrt{\lambda}R) - H_0^{(1)}(\sqrt{\lambda}R^*) \right] \]

\[ + \frac{1}{4\pi} \int_0^{2\sqrt{rr_0} \cos[(\varphi - \varphi_0)/2]} e^{i\sqrt{\lambda}(R^2 + v^2)^{1/2}} \frac{[R^2 + v^2]^{1/2}}{[R^2 + v^2]^{1/2}} dv \]

\[ - \frac{1}{4\pi} \int_0^{2\sqrt{rr_0} \cos[(\varphi + \varphi_0)/2]} e^{i\sqrt{\lambda}(R^*^2 + v^2)^{1/2}} \frac{[(R^*)^2 + v^2]^{1/2}}{[(R^*)^2 + v^2]^{1/2}} dv, \quad (7.216) \]
where, as in Figure 7.10,
\[ R^2 = r^2 + r_0^2 - 2rr_0 \cos(\varphi - \varphi_0), \]
\[ (R^*)^2 = r^2 + r_0^2 - 2rr_0 \cos(\varphi - (2\pi - \varphi_0)) = r^2 + r_0^2 - 2rr_0 \cos(\varphi + \varphi_0). \]

In view of the representation (7.202), we can finally write
\[
g(r, \varphi | r_0, \varphi_0) = \frac{1}{4\pi} \int_{-\infty}^{2\sqrt{rr_0} \cos[(\varphi - \varphi_0)/2]} e^{i\sqrt{\lambda}(R^2 + v^2)^{1/2}} \frac{1}{[R^2 + v^2]^{1/2}} dv - \frac{1}{4\pi} \int_{-\infty}^{2\sqrt{rr_0} \cos[(\varphi + \varphi_0)/2]} e^{i\sqrt{\lambda}((R^*)^2 + v^2)^{1/2}} \frac{1}{[(R^*)^2 + v^2]^{1/2}} dv. \] (7.217)

The result remains valid for \( \lambda \) real, positive.

**Plane-Wave Excitation**

If \( l \) is a fixed unit vector in 3-space and \( x \) is the position vector with respect to a fixed origin, it is easily seen that
\[ U = e^{i\omega x \cdot l} \]
is a solution of the homogeneous equation
\[ -\nabla^2 U - \omega^2 U = 0. \]

Affixing the time factor \( \exp(-i\omega t) \) to \( U \), we obtain a monochromatic solution of the wave equation; this solution has a constant value on the plane \( x \cdot l = t \) and travels with unit velocity in the \( l \) direction. Since the time factor \( \exp(-i\omega t) \) is always understood for the Helmholtz equation, we refer to \( \exp(i\omega x \cdot l) \) as a plane wave. Now let \( l \) point in the direction \( \pi + \varphi_0 \) in Figure 7.10. Then
\[ e^{i\omega x \cdot l} = e^{-i\omega r \cos(\varphi - \varphi_0)} \] (7.218)
represents a plane wave traveling in the \( (\pi + \varphi_0) \) direction, that is, emanating from the \( \varphi_0 \) direction. Such a plane wave can be considered as the limiting case of a cylindrical source being removed to infinity along the \( \varphi_0 \) direction with a suitable adjustment of strength.

At the finite observation point \( (r, \varphi) \), the field caused by a line source (cylindrical source) located at \( (r_0, \varphi_0) \) is
\[ u = \frac{i}{4} H_0^{(1)} \left\{ \omega \left[ r^2 + r_0^2 - 2rr_0 \cos(\varphi - \varphi_0) \right]^{1/2} \right\} \]
\[ = \frac{i}{4} H_0^{(1)} \left\{ \omega r_0 \left[ 1 + \left( \frac{r}{r_0} \right)^2 - 2 \left( \frac{r}{r_0} \right) \cos(\varphi - \varphi_0) \right]^{1/2} \right\}. \]

Since \( r_0 \) is large, we can write, approximately,
\[ u = \frac{i}{4} H_0^{(1)} \left\{ \omega r_0 - \omega r \cos(\varphi - \varphi_0) \right\}. \]
Using the asymptotic expansion of the Hankel function for large argument, see (B.13), Volume I,

\[ u \sim \frac{i}{4} \left[ \frac{2}{\pi \omega r_0} \right]^{1/2} e^{i\omega r_0} e^{-\omega r \cos(\varphi - \varphi_0)} e^{-i\omega/4}, \]

so that if we scale up our point source by the factor

\[ \frac{4}{i} e^{i\omega/4} \left[ \frac{\pi \omega r_0}{2} \right]^{1/2} e^{-i\omega r_0} = 4 e^{-i\omega/4} \left[ \frac{\pi \omega r_0}{2} \right]^{1/2} e^{-i\omega r_0} \tag{7.219} \]

and let \( r_0 \to \infty \), the point source becomes the plane wave (7.218).

We now return to the half-plane excited by a point source at \( (r_0, \varphi_0) \). Keeping \( \varphi_0 \) constant, multiplying the strength of the source by the scale factor (7.219), and letting \( r_0 \to \infty \), the incident field becomes a plane wave, and if we perform the same operations on (7.217), we shall obtain the corresponding solution of the half-plane problem. Carrying out this program, we find

\[
\begin{align*}
  u &= \frac{e^{-i\omega/4}}{\sqrt{\pi}} \left[ e^{-i\omega r \cos(\varphi - \varphi_0)} \int_{-\infty}^{\sqrt{2r_0} \cos(\varphi - \varphi_0)/2} e^{i\alpha^2} d\alpha ight] \\
  &\quad - e^{-i\omega r \cos(\varphi + \varphi_0)} \int_{-\infty}^{\sqrt{2r_0} \cos(\varphi + \varphi_0)/2} e^{i\alpha^2} d\alpha, \tag{7.220}
\end{align*}
\]

which is the form due to Sommerfeld.

From (7.220) we can deduce the high-frequency form of the field. With \( \omega \) large, and using the fact that

\[ \int_{-\infty}^{\infty} e^{i\alpha^2} d\alpha = e^{i\pi/4} \sqrt{\pi}, \]

we find the geometric optics field

\[
  u_G = \begin{cases} 
    e^{-i\omega r \cos(\varphi - \varphi_0)} - e^{-i\omega r \cos(\varphi + \varphi_0)} & \text{in region I;} \\
    e^{-i\omega r \cos(\varphi - \varphi_0)} & \text{in region II;} \\
    0 & \text{in region III.} \end{cases} \tag{7.221}
\]

This result follows from the observation that in region I both \( \cos[(\varphi - \varphi_0)/2] \) and \( \cos[(\varphi + \varphi_0)/2] \) are positive; in region II the first cosine is positive, the other negative; in region III both are negative (see Figure 7.11).

As expected, for high frequencies, in region II we only have the incoming wave, in region I we have both the incoming wave and a reflected wave, and region III is a shadow region. A similar analysis can be carried out for the field (7.217) due to a point source.
Time-Dependent Solution. An Instantaneous Line Source Diffracted by a Half-Plane

We turn our attention to the following problem for the wave equation. Referring again to Figure 7.10, we consider an instantaneous line source normal to the xy plane at \((r_0, \varphi_0)\). The field \(G(r, \varphi, t \mid r_0, \varphi_0, 0)\) is the causal Green's function for the wave equation in the region exterior to a half-plane. Therefore,

\[
\frac{\partial^2 G}{\partial t^2} - \nabla^2 G = \frac{\delta(r - r_0)\delta(\varphi - \varphi_0)\delta(t)}{r};
\]

\[G \equiv 0, \quad t < 0;\]

\[G = 0, \quad \varphi = 0, \quad \varphi = 2\pi.
\]

Taking a Laplace transform on time, we find

\[s^2 \tilde{G} - \nabla^2 \tilde{G} = \frac{\delta(r - r_0)\delta(\varphi - \varphi_0)}{r},\]

and, from (7.217),

\[
\tilde{G} = \frac{1}{4\pi} \int_{-\infty}^{2\sqrt{r_0}} \frac{e^{-s[R^2 + v^2]^{1/2}}}{[R^2 + v^2]^{1/2}} dv
- \frac{1}{4\pi} \int_{-\infty}^{2\sqrt{r_0}} \frac{e^{-s[(R^*)^2 + v^2]^{1/2}}}{[(R^*)^2 + v^2]^{1/2}} dv. \tag{7.222}
\]
Since the inverse of $e^{-sa}$ is $\delta(t - a)$, we find, for $t > 0$,

$$G = \frac{1}{4\pi} \int_{-\infty}^{2 \sqrt{r_0} \cos\left(\frac{\varphi - \varphi_0}{2}\right)} \frac{\delta(t - (R^2 + v^2)^{1/2})}{[R^2 + v^2]^{1/2}} \, dv$$

$$- \frac{1}{4\pi} \int_{-\infty}^{2 \sqrt{r_0} \cos\left(\frac{\varphi + \varphi_0}{2}\right)} \frac{\delta(t - [(R^*)^2 + v^2]^{1/2})}{[(R^*)^2 + v^2]^{1/2}} \, dv.$$

As a preliminary observation, we remark that the first integral surely vanishes for $t < R$ and the second for $t < R^*$. The argument of the delta function in the first integrand vanishes at

$$v = \pm \sqrt{t^2 - R^2},$$

and each zero included in the range of integration contributes $(t^2 - R^2)^{-1/2}$. For the second integrand the critical points are

$$v = \pm \sqrt{t^2 - (R^*)^2},$$

each of which contributes $[t^2 - (R^*)^2]^{-1/2}$ to the integral.

It remains only to decide whether these zeros are to be included in the range of integration. For the first integral, the inequality

$$\sqrt{t^2 - R^2} < 2\sqrt{r_0} \cos \frac{\varphi - \varphi_0}{2}$$

is possible only if $t < r + r_0$ and $\varphi$ is in region I or II [see Figure 7.1, where we now have a line source at $(r_0, \varphi_0)$ and no plane wave]. The inequality

$$-\sqrt{t^2 - R^2} < 2\sqrt{r_0} \cos \frac{\varphi - \varphi_0}{2}$$

is true whenever $\varphi$ is in region I or II or if $\varphi$ is in region III and $t > r + r_0$.

For the second integral,

$$\sqrt{t^2 - (R^*)^2} < 2\sqrt{r_0} \cos \frac{\varphi + \varphi_0}{2}$$

can occur only if $t < r + r_0$ and $\varphi$ is in region I, whereas the inequality

$$-\sqrt{t^2 - (R^*)^2} < 2\sqrt{r_0} \cos \frac{\varphi + \varphi_0}{2}$$

is true wherever $\varphi$ is in region I, or if $t > r + r_0$ and $\varphi$ is in region II or III.

Collecting the results we find, with $H$, the Heaviside function,
G(r, \phi, t \mid r_0, \phi_0, 0) =
\begin{align*}
&\begin{cases}
\frac{H(t - R)}{2\pi\sqrt{t^2 - R^2}} - \frac{H(t - R^*)}{2\pi\sqrt{t^2 - (R^*)^2}}, & \text{\[7.223\]} \quad \phi \text{ in region I, } t < r + r_0; \\
\frac{H(t - R)}{2\pi\sqrt{t^2 - R^2}}, & \phi \text{ in region II, } t < r + r_0; \\
0, & \phi \text{ in region III, } t < r + r_0; \\
\frac{1}{4\pi}\left[\frac{1}{\sqrt{t^2 - R^2}} - \frac{1}{\sqrt{t^2 - (R^*)^2}}\right], & \text{all } \phi, \ t > r + r_0.
\end{cases}
\end{align*}

Since for any (r, \phi) we have r + r_0 > R and r + r_0 > R^*, we can rewrite the expression for G as follows.

In region I:
\begin{align*}
G = \begin{cases}
0 < t < R; \\
\frac{1}{2\pi\sqrt{t^2 - R^2}}, & R < t < R^*; \\
\frac{1}{2\pi\sqrt{t^2 - R^2}} - \frac{1}{2\pi\sqrt{t^2 - (R^*)^2}}, & R^* < t < r + r_0; \\
\frac{1}{4\pi\sqrt{t^2 - R^2}} - \frac{1}{4\pi\sqrt{t^2 - (R^*)^2}}, & t > r + r_0.
\end{cases}
\end{align*}

In region II:
\begin{align*}
G = \begin{cases}
0, & 0 < t < R; \\
\frac{1}{2\pi\sqrt{t^2 - R^2}}, & R < t < r + r_0; \\
\frac{1}{4\pi\sqrt{t^2 - R^2}} - \frac{1}{4\pi\sqrt{t^2 - (R^*)^2}}, & t > r + r_0.
\end{cases}
\end{align*}

In region III:
\begin{align*}
G = \begin{cases}
0, & 0 < t < r + r_0; \\
\frac{1}{4\pi\sqrt{t^2 - R^2}} - \frac{1}{4\pi\sqrt{t^2 - (R^*)^2}}, & t > r + r_0.
\end{cases}
\end{align*}

The physical interpretation is clear. In region I the primary wave begins to be felt at \( t = R \); then at the later time \( t = R^* \), the reflected wave appears; finally the diffracted wave arrives at \( t = r + r_0 \). In region II, there is no

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reflected wave and in region III (the shadow region) neither a reflected nor a primary wave. It is remarkable that after all waves have made their appearance, the total field has the same form in all regions and that this form is just half the field due to the primary source and image source in free space.

**EXERCISES**

7.31 *Addition theorem for spherical waves.* For \( \lambda \) not in \([0, \infty)\), consider

\[
-\nabla^2 u - \lambda u = \frac{\delta(r - r_0)\delta(\theta)}{2\pi r^2 \sin \theta},
\]

where \( r \) and \( \theta \) are spherical coordinates. From (5.31), this means that there is a point source on the positive \( z \) axis at a distance \( r_0 \) from the origin. By expanding in the \( \theta \) eigenfunctions \( P_n(\cos \theta) \), show that

\[
u = \frac{e^{i\sqrt{\lambda} R}}{4\pi R} = \frac{i}{8\sqrt{rr_0}} \sum_{n=0}^{\infty} (2n + 1)P_n(\cos \theta)J_{n+1/2}(\sqrt{\lambda} r) \times H_{n+1/2}^{(1)}(\sqrt{\lambda} r),
\]

(7.224)

where

\[
R^2 = r^2 + r_0^2 - 2rr_0 \cos \theta,
\]

and \( \lambda \) has positive imaginary part. Using the principle of limiting absorption, show that (7.224) still holds for \( \lambda \) real, positive. By choosing \( \lambda = -k^2, k > 0 \), derive

\[
e^{-kR} = \frac{1}{4\pi R} \sum_{n=0}^{\infty} (2n + 1)P_n(\cos \theta)I_{n+1/2}(kr)K_{n+1/2}(kr).
\]

(7.225)

Show that the correct limit is achieved as \( r_0 \to 0 \).

7.32 Obtain expressions similar to those of Exercise 7.31 when a ring of sources of total strength \( 1 \) is located on \( r = r_0, \theta = \theta_0 \).

7.33 Find the temperature in \( r > a \), where \( r \) is a spherical coordinate, if the boundary is at zero temperature and the initial temperature corresponds to an instantaneous surface source density on the sphere \( r = r_0 \); that is,

\[
u(r, 0) = \frac{1}{4\pi r^2} \delta(r - r_0), \quad r_0 > a.
\]

7.34 Consider the Green’s function for \(-\nabla^2 - \lambda I\) in the exterior of the circle \( r = a \) with vanishing boundary condition. (a) Write

\[
g = E + v,
\]

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where \( E \) is the free-space fundamental solution (7.154). If \( \lambda \) is not in \([0, \infty)\), \( v \) must vanish at \( r = \infty \) and has the representation

\[
v = \sum_{n=-\infty}^{\infty} v_n H_n^{(1)}(\sqrt{\lambda} r)e^{in\varphi}.
\]

Calculate \( v_n \) by using the addition theorem (7.163).

(b) Find \( g \) by expanding directly in the set \( e^{in\varphi} \) and solving the resulting ordinary differential equation in the radial direction. Reconcile your answer with that in part (a).

7.35 Solve Exercise 7.34 by a radial expansion. Consider only the case \( \lambda = -k^2, k > 0. \) The \( v \) spectrum of the radial equation (7.161) is discrete. Obtain the normalized eigenfunctions by the method of Exercise 4.8.

7.36 Consider the Helmholtz equation in the slab

\[-\infty < x < \infty, \quad -\infty < y < \infty, \quad 0 < z < h,
\]

with vanishing normal derivative on the faces \( z = 0 \) and \( z = h \). A ring of sources is placed at \( z = z_0, \ x^2 + y^2 = r_0^2 \). Find the response by using a Hankel transform on \( r \) and also by an expansion in the \( z \) eigenfunctions.

7.37 Find the steady temperature \( u(r, z) \) in the slab

\[-\infty < x < \infty, \quad -\infty < y < \infty, \quad 0 < z < h,
\]

when \( u(r, 0) = 0, \ u(r, h) = f(r). \) Here \( r \) is the usual polar coordinate.

7.38 Point source in a cylindrical wave guide. Consider an infinite cylinder of cross section \( R \), the generators of the cylinder being parallel to the \( z \) axis. The homogeneous Helmholtz equation is

\[\nabla_3^2 u + \lambda u = 0, \quad x, y \text{ in } R, \quad -\infty < z < \infty,
\]

where \( \nabla_3^2 \) is the three-dimensional Laplacian in \((x, y, z)\). Taking the boundary condition to be \( u = 0 \) on the lateral surface of the cylinder, and writing

\[u(x, y, z) = Z(z)\varphi(x, y),\]

we find

\[-Z'' = \mu Z, \quad -\infty < z < \infty,
\[-\nabla_2^2 \varphi = v\varphi, \quad x, y \text{ in } R, \quad \text{(7.226)}
\]

where

\[v = \lambda - \mu, \quad \nabla_2^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.
\]
In this separation $\lambda$ is fixed and either $\nu$ or $\mu$ may be taken as the eigenvalue parameter. With (7.226) is associated the boundary condition $\varphi = 0$ on $C$, the closed curve bounding $R$. This transverse problem then leads to the set of two-dimensional eigenfunctions $\varphi_n(x, y)$ and eigenvalues $\nu_n$, $n = 1, 2, \ldots$.

Now consider the Green’s function for this cylinder. Let a unit source be placed at the interior point $(x_0, y_0, 0)$. By expanding in the transverse eigenfunctions $\varphi_n(x, y)$, obtain

$$
g(x, y, z | x_0, y_0, 0) = \sum_{n=1}^{\infty} \frac{e^{-\sqrt{\nu_n - \lambda}|z|}}{2\sqrt{\nu_n - \lambda}} \varphi_n(x, y) \overline{\varphi}_n(x_0, y_0), \quad (7.227)
$$

where $\sqrt{\nu_n - \lambda}$ is the square root with positive real part (a choice which is always possible when $\lambda$ is not real and positive). By the principle of limiting absorption, we find that (7.227) is still valid for $\lambda$ real, positive, if we set

$$
\sqrt{\nu_n - \lambda} = \begin{cases} |v_n - \lambda|^{1/2}, & \text{for } n \text{ such that } v_n \geq \lambda, \\
-i|\lambda - v_n|^{1/2}, & \text{for } n \text{ such that } v_n < \lambda.
\end{cases}
$$

In particular, if $0 < \lambda < \nu_1$, all terms in (7.227) are still exponentially decaying. The expression (7.227) is correct even for $\lambda = 0$, when it gives the Green’s function for the negative Laplacian.

Using instead an axial expansion (Fourier transform on $z$), obtain

$$
g = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iax} g^\wedge \, dx,
$$

where

$$
-\nabla_z^2 g^\wedge - (\lambda - \alpha^2) g^\wedge = \delta(x - x_0)\delta(y - y_0), \quad x, y \text{ in } R;
$$

$$
g^\wedge = 0, \quad x, y \text{ on } C.
$$

Derive the double expansion

$$
g(x, y, z | x_0, y_0, 0) = \sum_{n=1}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-iax} \varphi_n(x, y) \overline{\varphi}_n(x_0, y_0)}{v_n + \alpha^2 - \lambda} \, dx.
$$

7.39 In the special case of a line source located in front of a half-plane, that is, $\varphi_0 = \pi$ in (7.206), obtain from (7.217) the simpler form

$$
g = \frac{1}{2\pi} \int_{0}^{2\sqrt{r r_0} \sin(\varphi/2)} e^{\sqrt{\lambda} [r^2 + r_0^2 + 2rr_0 \cos \varphi + v^2]^{1/2}} \, dv.
$$

Show that the total current on the half-plane,

$$
I(r) = \left( \frac{1}{r} \frac{\partial g}{\partial \varphi} \right)_{\varphi=0^+} - \left( \frac{1}{r} \frac{\partial g}{\partial \varphi} \right)_{\varphi=2\pi^-},
$$
is given by
\[
\frac{1}{\pi} \left( \frac{r_0}{r} \right)^{1/2} e^{i\sqrt{2}(r+r_0)} \frac{e^{i\alpha}}{r + r_0},
\]
(7.228)
which exhibits the characteristic singularity \( r^{-1/2} \) at the edge.

7.40 For diffraction of a plane wave by a half-plane, show from (7.220) that the total current \( I \) on the half-plane is
\[
\frac{2e^{-i\pi/4}}{r\sqrt{\pi}} \left[ e^{i\alpha r} \sqrt{2r_0} \sin \frac{\phi_0}{2} - 2i\omega r \sin \phi_0 e^{-i\alpha r} \cos \phi_0 \int_0^{\sqrt{2r_0} \cos(\phi_0/2)} e^{ix^2} dx \right].
\]
(7.229)
For normal incidence, \( \phi_0 = \pi/2 \), and
\[
I = \frac{2e^{-i\pi/4}}{r\sqrt{\pi}} \left[ e^{i\alpha r} \sqrt{r_0} - 2i\omega r \int_0^{\sqrt{r_0}} e^{ix^2} dx \right].
\]
(7.230)

7.41 By the method of Section 7.13, find the response for a line source exciting a half-plane when the boundary condition is \( \partial g/\partial n = 0 \) on the half-plane. Consider also the case of an incoming plane wave.

7.42 Consider the propagation of sound waves in a composite medium. If the speed of sound is \( c_1 \) in medium 1 and \( c_2 \) in medium 2, the velocity potential \( u(x, t) \) satisfies
\[
\frac{\partial^2 u}{\partial t^2} - c_1^2 \nabla^2 u = q(x, t), \quad x \text{ in medium } 1,
\]
\[
\frac{\partial^2 u}{\partial t^2} - c_2^2 \nabla^2 u = q(x, t), \quad x \text{ in medium } 2.
\]

At the interface the normal velocity \( \partial u/\partial n \) and the pressure are continuous. Since the pressure is \( -\rho(\partial u/\partial t) \), where \( \rho \) is the density, we must have \( \rho_1(\partial u/\partial t) = \rho_2(\partial u/\partial t) \) at the interface.

Let medium 1 be the half-space \( z < 0 \). Specializing to the case of a monochromatic point source located at \((0, 0, z_0)\) in the first medium, we find that the complex potential \( U \) satisfies
\[
-\nabla^2 U - k_1^2 U = \delta(z - z_0)\delta(x)\delta(y), \quad z, z_0 > 0, \quad -\infty < x, y < \infty;
\]
\[
-\nabla^2 U - k_2^2 U = 0, \quad z < 0, \quad -\infty < x, y < \infty,
\]
where \( k_1^2 = \omega^2/c_1^2 \), \( k_2^2 = \omega^2/c_2^2 \), and the boundary conditions are
\[
\rho_1 U(x, y, 0+) = \rho_2 U(x, y, 0-); \quad \frac{\partial U}{\partial z}(x, y, 0+) = \frac{\partial U}{\partial z}(x, y, 0-).
\]

Apply a Hankel transform to the radial polar coordinate to solve for \( U \).
7.43 Consider the three-dimensional problem

\[-\nabla^2 u + k^2 u = q(x), \quad |x| < 1, \quad u = 0 \text{ on } |x| = 1,\]

where \(k > 0\). Show that

\[u(0) = \int_{|\xi| < 1} \left[ e^{-k|\xi|} - \frac{e^{-k}}{\sinh k |\xi|} \right] \frac{q(\xi)}{4\pi|\xi|} d\xi.\]

[Hint: Express \(u(0)\) in terms of the Green’s function; then use the symmetry of the Green’s function. Show that the result remains correct when \(k = 0\).]

7.14 REPRESENTATION OF SOLUTIONS OF THE HELMHOLTZ EQUATION IN EXTERIOR DOMAINS

With \(\lambda\) a fixed complex number, consider a function which satisfies

\[\nabla^2 u + \lambda u = 0\]

outside a bounded region in three dimensions. With a fixed but arbitrary center, there must exist a sphere of finite radius (say, \(r_0\)) outside which the above equation is satisfied. For \(r > r_0\), we may expand \(u\) in the complete set of spherical harmonics \(Y^m_n(\theta, \varphi)\), which are the angular functions arising from separation of the Helmholtz equation in spherical coordinates. We then write

\[u(r, \theta, \varphi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} u_{m,n}(r) Y^m_n(\theta, \varphi), \quad r > r_0,\]

with

\[u_{m,n}(r) = \frac{1}{N_{m,n}} \int_0^{2\pi} d\varphi \int_0^\pi d\theta \sin \theta \ u(r, \theta, \varphi) Y^m_n(\theta, \varphi),\]

where the normalization constants \(N_{m,n}\) are given explicitly in (A.5). The differential equation for \(u\) can be written

\[
\frac{\partial}{\partial r} \left( r^2 \frac{\partial u}{\partial r} \right) - \frac{1}{\sin \theta} Su + \lambda r^2 u = 0, \quad (7.231)
\]

where

\[S = -\frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) - \frac{1}{\sin \theta} \frac{\partial^2}{\partial \varphi^2},\]

\[SY^m_n = n(n + 1) \sin \theta \ Y^m_n, \quad (7.232)\]

Multiplying (7.231) by

\[
\frac{\sin \theta \ Y^m_n(\theta, \varphi)}{N_{m,n}},
\]
and integrating over $\theta$ and $\varphi$, we obtain
\[
(r^2u'_{m,n})' - n(n+1)u_{m,n} + \lambda r^2u_{m,n} = 0, \tag{7.233}
\]
where the prime denotes differentiation with respect to $r$. Writing
\[
u_{m,n} = r^{-1/2}v_{m,n},
\]
we find that $v_{m,n}$ satisfies the ordinary differential equation
\[
(rv'_{m,n})' + \lambda rv_{m,n} - (n + 1/2)^2 \frac{v_{m,n}}{r} = 0,
\]
which is Bessel's equation of order $n + 1/2$ with parameter $\lambda$. The general solution of (7.233) can therefore be written as a linear combination of the functions
\[
r^{-1/2}H_{n+1/2}^{(1)}(\sqrt{\lambda} r), \quad r^{-1/2}H_{n+1/2}^{(2)}(\sqrt{\lambda} r), \tag{7.234}
\]
where, as usual,
\[
\sqrt{\lambda} = |\lambda|^{1/2} e^{i\psi/2}, \quad 0 \leq \psi < 2\pi.
\]
If $\lambda$ is not in $[0, \infty)$, then only the first of the two functions in (7.234) is bounded at infinity; in fact, the Hankel function of the first kind decays exponentially at infinity, whereas the Hankel function of the second kind grows exponentially at infinity. Thus if we are looking for a solution $u$ of (7.233) which is bounded at infinity, we can write
\[
u = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{m,n} Y_m^n(\theta, \varphi) H_{n+1/2}^{(1)}(\sqrt{\lambda} r)/r^{1/2}, \quad r > r_0 \tag{7.235}
\]
and this solution automatically has exponential decay at infinity. Pursuing this point further, we observe that from the asymptotic behavior of each of the Hankel functions, we have the following asymptotic representation for $u$:
\[
u \sim f(\theta, \varphi) e^{i\sqrt{\lambda} r}/r. \tag{7.236}
\]
Since
\[
\sqrt{\lambda} = \alpha + i\beta, \quad \beta > 0,
\]
u decays as $e^{-\beta r}/r$ at infinity.

It should be observed that the successive Hankel functions appearing in (7.235) all exhibit the same decay at infinity. This is in sharp contrast with the representation for bounded harmonic functions ($\lambda = 0$),
\[
u = a_0 + \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{m,n} Y_m^n(\theta, \varphi)r^{-n-1},
\]
where the successive terms are of decreasing order in $r$ (although not exponentially decaying, of course!).

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If $\lambda$ is real, positive, $\lambda = \omega^2$, $\omega > 0$, the argument which led to (7.235) fails. Both the solutions (7.234) are bounded at infinity, although neither is exponentially decaying. To pick out the physically relevant solution we appeal to the principle of limiting absorption, which again yields the representation (7.235) with $\sqrt{\lambda} = \omega$. Alternatively we observe that a term like
\[ \frac{e^{i\sqrt{\lambda}r}}{r} \]
represents an outgoing spherical wave (when the time factor $e^{-iot}$ is affixed), whereas
\[ \frac{e^{-i\sqrt{\lambda}r}}{r}, \]
which characterizes the asymptotic behavior of Hankel functions of the second kind, would describe an incoming spherical wave from infinity and must therefore be rejected.

Bounded solutions of the two-dimensional homogeneous Helmholtz equation in the exterior of a circle of radius $r_0$ can be represented in the form
\[ u = \sum_{n=-\infty}^{\infty} a_n e^{i\sigma} H_n^{(1)}(\sqrt{\lambda} r), \quad r > r_0, \quad (7.237) \]
for all $\lambda \neq 0$. Using the asymptotic representation of Hankel functions, we see that
\[ u \sim f(\sigma) e^{i\sqrt{\lambda}r} \frac{1}{r^{1/2}}, \quad r \to \infty. \quad (7.238) \]
which describes an outgoing cylindrical wave. If $\lambda$ is not in $[0, \infty)$, we have the exponential decay characteristic of the damped wave equation. If $\lambda$ is real and positive there is no exponential decay but (7.238) still describes an outgoing cylindrical wave.

**Radiating Exterior Dirichlet Problem**

Let $R$ be the exterior of a bounded region (in two or three dimensions) with smooth boundary $\sigma$. The radiating exterior problem is
\[ \nabla^2 u + \lambda u = 0, \quad x \text{ in } R, \quad u \text{ outgoing at } \infty, \quad u = f(x), \quad x \text{ on } \sigma, \quad (7.239) \]
where $\lambda$ and $f(x)$ are given.

The statement that $u$ is outgoing at $\infty$ means that for $\lambda$ not in $[0, \infty)$, $u$ should decay exponentially at infinity, uniformly with respect to the angular coordinates. If $\lambda$ is real, $0 < \lambda < \infty$, the outgoing condition can roughly be interpreted as (7.236) in three dimensions or (7.238) in two dimensions, but for our purposes it is enough to use the principle of limiting absorption.
We now prove uniqueness of the solution of (7.239) in the case \( \lambda = a + ib, b \neq 0 \). Uniqueness means that there cannot be two solutions of the problem (7.239). Of course one can always achieve uniqueness by imposing such severe restrictions on \( u \) that no solution exists! This is certainly not the goal of a uniqueness theorem; we should impose conditions severe enough so that not more than one solution exists but mild enough so that the desired physical solution survives. The discussion of existence of a solution of (7.239) is deferred to Section 7.15.

Now suppose \( u_1 \) and \( u_2 \) are two solutions of (7.239); then \( v = u_1 - u_2 \) satisfies

\[
\nabla^2 v + \lambda v = 0, \quad x \text{ in } R; \quad v \text{ outgoing at } \infty; \quad v = 0, \quad x \text{ on } \sigma. \tag{7.240}
\]

In view of the assumption on \( \lambda \), \( v \) decays exponentially at infinity, and so therefore does its complex conjugate \( \bar{v} \), which satisfies

\[
\nabla^2 \bar{v} + \bar{\lambda} \bar{v} = 0, \quad x \text{ in } R; \quad \bar{v} \text{ outgoing at } \infty; \quad \bar{v} = 0, \quad x \text{ on } \sigma. \tag{7.241}
\]

If we multiply (7.240) by \( \bar{v} \) and (7.241) by \( v \), subtract, and integrate over a region \( R \), bounded internally by \( \sigma \) and externally by a large sphere \( \sigma_r \) of radius \( r \), we find

\[
\int_{R_r} (\bar{v} \nabla^2 v - v \nabla^2 \bar{v}) dx = (\bar{\lambda} - \lambda) \int_{R_r} |v|^2 \, dx,
\]

or

\[
(\bar{\lambda} - \lambda) \int_{R_r} |v|^2 \, dx = \int_{\sigma_r} \left( \frac{\bar{v} \partial v}{\partial n} - v \frac{\partial \bar{v}}{\partial n} \right) dS - \int_{\sigma} \left( \frac{\partial v}{\partial n} - v \frac{\partial \bar{v}}{\partial n} \right) dS,
\]

where \( n \) is shown in Figure 7.12. In view of the outgoing condition at infinity, the surface integral on \( \sigma_r \) vanishes as \( r \to \infty \). The integral on \( \sigma \) is 0 by the boundary condition, so that

\[
\lim_{r \to \infty}(\bar{\lambda} - \lambda) \int_{R_r} |v|^2 \, dx = 0.
\]

Since \( \bar{\lambda} - \lambda \neq 0 \), we conclude that \( v \equiv 0 \) in \( R \), and hence that (7.239) has at most one solution.

Remarks. 1. If \( \lambda = -k^2, k > 0 \), the proof is easily modified. Multiply (7.240) by \( \bar{v} \) and apply Green’s theorem (6.1a). Again we then find \( v \equiv 0 \).

2. If \( \lambda = \omega^2, \omega > 0 \), uniqueness follows again from the principle of limiting absorption. On the other hand, we may prefer to characterize the boundary condition at \( \infty \) in (7.239) directly without appealing to a complex \( \lambda \); this can be accomplished by requiring that the Rellich-Sommerfeld radiation condition,

\[
\lim_{r \to \infty} \int_{\sigma_r} \left| \frac{\partial u}{\partial r} - i \omega u \right|^2 dS = 0, \tag{7.242}
\]

be satisfied.
To show that (7.242) really characterizes radiation, it suffices to show that the energy flux (outward) through $\sigma_r$ is positive. From (7.148) we find that the energy flux $P_r$ through $\sigma_r$ is

$$P_r = \frac{\omega i}{4} \int_{\sigma_r} \left( u \frac{\partial \bar{u}}{\partial n} - \bar{u} \frac{\partial u}{\partial n} \right) dS.$$ 

From (7.242) we have

$$\lim_{r \to \infty} \int_{\sigma_r} \left[ \left| \frac{\partial u}{\partial r} \right|^2 + \omega^2 |u|^2 + i\omega \left( \bar{u} \frac{\partial u}{\partial n} - u \frac{\partial \bar{u}}{\partial n} \right) \right] dS = 0.$$ 

Therefore,

$$\lim_{r \to \infty} P_r \geq 0,$$

so that we have, in fact, outward energy flux. Furthermore, one can show that (7.240) with $\lambda = \omega^2$ and with $v$ required to satisfy (7.242) has at most one solution.

3. For two dimensions we merely change the word "sphere" to "circle" and the proof proceeds in exactly the same way.

4. Let $R$ be the exterior of a region whose boundary $\sigma$ is not smooth. A typical example of a two-dimensional case is shown in Figure 7.13.

At the vertex $A$, the solution of (7.239) or its gradient may have a singularity. This phenomenon has already been observed for potential theory (see Exercise 6.10, for instance) and persists for the Helmholtz equation. These singularities at $A$ prevent us from applying Green's
theorem directly to $R$, and we must instead use a region whose interior boundary $\sigma'$ is the dashed curve in Figure 7.13. The proof that (7.240) has only the zero solution carries over if the quantity
\[
\int_{\sigma'} \left( v \frac{\partial \bar{v}}{\partial n} - \bar{v} \frac{\partial v}{\partial n} \right) dS = 2i \text{ Im} \int_{\sigma'} v \frac{\partial \bar{v}}{\partial n} dS \tag{7.243}
\]
tends to 0 as $\sigma'$ shrinks to $\sigma$. Since $v$ and $\partial v/\partial n$ are well behaved except in the neighborhood of $A$, and $v$ vanishes on $\sigma$, it suffices to require that
\[
\text{Im} \int_{\sigma''} v \frac{\partial \bar{v}}{\partial n} dS
\]
approach 0 as $\sigma''$ (a small circle with center at $A$) shrinks to $A$. We would like to ensure that this condition is satisfied by the difference of two solutions of (7.239). This can be accomplished by imposing on the solution of (7.239) the additional requirement that $|u|^2 + |\text{grad } u|^2$ be locally integrable. Physically this means that there is finite energy in any bounded region of space and that any singularity of the field at the vertex $A$ is sufficiently weak so that no source is concentrated there. This condition will always be associated implicitly with any exterior problem involving a nonsmooth boundary.

5. If the boundary condition on $\sigma$ in (7.239) is specified in terms of $\partial u/\partial n$ or $\partial u/\partial n + ax$, where $\alpha$ is real, the uniqueness proof still holds.

6. If $R$ is an infinite wedge-shaped region (not the exterior of a bounded region), the proof of uniqueness is more difficult and such cases are best treated on an ad hoc basis.

7.15 SCATTERING PROBLEM

We shall consider the scattering of incident fields by obstacles in two or three dimensions. First we must explain the term incident field. By this we mean a solution of either the homogeneous or inhomogeneous Helmholtz equation in the whole space. Typically the incident field is generated either
by concentrated sources located in the finite portion of space or by sources at infinity (as would be the case for a plane wave $e^{i\sqrt{\lambda}x \cdot \alpha}$ traveling in the \( \alpha \) direction). An incident field due to point sources is outgoing at infinity since it is a superposition of outgoing spherical or cylindrical waves. On the other hand, a plane wave is not outgoing because it behaves like $e^{-i\sqrt{\lambda}r}$ in the direction of incidence $-\alpha$.

We now introduce a bounded obstacle in a source-free region of the incident field. The interior of the obstacle is denoted by $R^*$, its exterior by $R$ and its boundary by $\sigma$. The field $u_i$ will be distorted to satisfy the boundary condition on $\sigma$ which we take as $u = 0$. The total field can be written

$$u = u_i + u_s,$$  
(7.244)

where $u_s$ is a scattered field defined only in the exterior region $R$. We can regard $u_s$ as being generated by sources induced on the surface $\sigma$. As such $u_s$ will be outgoing at infinity and will satisfy the homogeneous Helmholtz equation in the exterior region $R$. Thus we have the following radiating exterior Dirichlet problem for $u_s$:

$$\nabla^2 u_s + \lambda u_s = 0, \quad x \text{ in } R, \quad u_s \text{ outgoing at } \infty,$$

$$u_s = -u_i, \quad x \text{ on } \sigma.$$  
(7.245)

Since $u_i$ is known everywhere in space, the value of $u_s$ is known on $\sigma$. If the boundary $\sigma$ has edges, we must also impose a restriction on the allowed singularities of $u_s$ at the edges so that they will not radiate energy.

Next we shall express $u_s$ as an integral on $\sigma$ involving as the only unknown the normal derivative of the total field $u$ on $\sigma$.

The free-space fundamental solution $E(x \mid \xi)$ of the Helmholtz equation satisfies

$$-\nabla^2 E - \lambda E = \delta(x - \xi), \quad \text{for all } x, \xi.$$  
(7.246)

The explicit form for $E$ in two and three dimensions is given by (7.154) and (7.152), respectively. We multiply equation (7.245) by $E$, and (7.246) by $u_s$, add, and integrate over the region $R_r$, bounded internally by $\sigma$ and externally by a large sphere $\sigma_r$. Then, if $\xi$ is in $R_r$,

$$u_s(\xi) = \int_{R_r} (E \nabla^2 u_s - u_s \nabla^2 E) \, dx$$

$$= \int_{\sigma_r} \left( E \frac{\partial u_s}{\partial n} - u_s \frac{\partial E}{\partial n} \right) dS_x + \int_{\sigma} \left( u_s \frac{\partial E}{\partial n} - E \frac{\partial u_s}{\partial n} \right) dS_x.$$  

As $r \to \infty$, the integral over $\sigma_r$ tends to zero since $E$ and $u_s$ are exponentially decaying at $\infty$. Therefore, since $u_s = u - u_i$ and $u$ vanishes on $\sigma$,

$$u_s(\xi) = \int_{\sigma} \left( E \frac{\partial u_i}{\partial n} - u_i \frac{\partial E}{\partial n} \right) dS_x - \int_{\sigma} E \frac{\partial u}{\partial n} dS_x, \quad \xi \text{ in } R.$$  
(7.247)
To evaluate the first surface integral, observe that

$$\nabla^2 u_i + \lambda u_i = 0, \quad x \text{ in } R^*, \quad (7.248)$$

since $R^*$ is source-free for the incident field. Multiply (7.246) by $u_i$, (7.248) by $E$, add, and integrate over $R^*$ to find

$$\int_S \left( E \frac{\partial u_i}{\partial n} - u_i \frac{\partial E}{\partial n} \right) dS_x = \begin{cases} 0, & \xi \text{ in } R; \\ u_i(\xi), & \xi \text{ in } R^*. \end{cases}$$

In (7.247), $\xi$ is in $R$, therefore

$$u_s(\xi) = -\int_S E(x \mid \xi) I(x) dS_x,$$

where

$$I(x) = \frac{\partial u}{\partial n}(x), \quad x \text{ on } \sigma.$$}

Relabeling variables and using the fact that

$$E(x \mid \xi) = E(\xi \mid x),$$

we obtain

$$u_s(x) = -\int_S E(x \mid \xi) I(\xi) dS_\xi, \quad x \text{ in } R. \quad (7.249)$$

Thus the scattered field has been expressed in all of $R$ in terms of the unknown $I$ on $\sigma$. In the electromagnetic case $I$ is the current on the obstacle. If $I$ can be determined, the problem has been solved. To this end let the point $x$ approach a point $s$ on the boundary; in view of the boundary condition satisfied by $u_s$, the left side approaches $-u_i(s)$. The right side can be considered as a simple-layer potential for the Helmholtz equation; by methods similar to those used for potential theory, it can be shown that such potentials are continuous on $\sigma$. Therefore we obtain the integral equation

$$u_s(s) = \int_S E(s \mid \xi) I(\xi) dS_\xi, \quad s \text{ on } \sigma, \quad (7.250)$$

to determine $I$.

The general procedure is then to solve this integral equation for $I$ and then substitute in (7.249) to calculate $u_s(x)$ at all points in $R$. The total field $u$ is equal to $u_s(x)$ plus the known incident field $u_i(x)$. Instead of the integral equation of the first kind (7.250), it is possible to derive an integral equation of the second kind for $I$ (see Exercise 7.44).

If the obstacle $\sigma$ is an open thin shell (see Figure 6.5), formula (7.249) and the integral equation (7.250) remain valid as long as $I$ is interpreted as the total current on $\sigma$; that is,

$$I(x) = I_+(x) + I_-(x) = \frac{\partial u}{\partial n_+}(x) + \frac{\partial u}{\partial n_-}(x), \quad x \text{ on } \sigma,$$
where \( n_+ \) and \( n_- \) are normals on either side of \( \sigma \) (therefore, at \( x \), \( n_+ \) and \( n_- \) are in opposite directions). In Exercise 7.45 we show how to obtain the separate values of \( I_+ \) and \( I_- \) on either side of \( \sigma \).

### Plane Wave Excitation

We set \( \lambda = \omega^2 \), \( \omega \) real, positive, and consider the incident field

\[
    u_i = e^{i\omega \cdot x \cdot a};
\]

that is, the incident field is a plane wave traveling in the \( \alpha \) direction and emanating from the \(-\alpha\) direction at infinity. This plane wave excites a current \( I_\alpha(x) \) on the surface of the obstacle, where the subscript reminds us of the direction of propagation of the incident wave. According to (7.249), the scattered field is

\[
    u_s(x) = -\int_\sigma E(x \mid \xi)I_\alpha(\xi)dS_\xi,
\]

where, for three-dimensional problems,

\[
    E(x \mid \xi) = \frac{e^{i\omega |x - \xi|}}{4\pi|x - \xi|}.
\]

The scattered field at large distances from the obstacle must behave as an outgoing spherical wave. This is confirmed by substituting the asymptotic expression for \( E \) in the formula for \( u_s \). We have

\[
    e^{i\omega |x - \xi|} = e^{i\omega |x|(|1 + |\xi|^2/|x|^2 - 2(\cos \psi)|\xi|/|x|)|^1/2},
\]

where \( \psi \) is the angle between the vectors \( x \) and \( \xi \). Since \( \xi \) is restricted to \( \sigma \) and \( x \) is in the far field, \(|\xi|/|x|\) is small, and

\[
    e^{i\omega |x - \xi|} \sim e^{i\omega |x|}e^{-i\omega |\xi| \cos \psi}.
\]

If we introduce a unit vector \( \beta \) to identify a particular direction in the far field, we have

\[
    \frac{e^{i\omega |x - \xi|}}{4\pi|x - \xi|} \sim \frac{e^{i\omega |x|}e^{-i\omega \cdot \xi \cdot \beta}}{4\pi|x|},
\]

which is uniformly valid for all \( \xi \). Substituting in (7.251), we find

\[
    u_s(x) \sim -\frac{e^{i\omega |x|}}{4\pi|x|} \int_\sigma e^{-i\omega \cdot \xi \cdot \beta}I_\alpha(\xi)dS_\xi,
\]

which shows that \( u_s(x) \) behaves like a spherical wave with a far-field amplitude

\[
    A(\beta, \alpha) = -\frac{1}{4\pi} \int_\sigma e^{-i\omega \cdot \xi \cdot \beta}I_\alpha(\xi)dS_\xi.
\]
Here \( A(\beta, \alpha) \) is the far-field amplitude in the observation direction \( \beta \) when the incident wave travels in the \( \alpha \) direction. We observe, from (7.253), that

\[
A(-\alpha, -\beta) = -\frac{1}{4\pi} \int_{\sigma} e^{i\omega \xi \cdot \alpha} I_{-\beta}(\xi) dS_{\xi}.
\]

From (7.250) we see that \( I_{-\beta} \) and \( I_{\alpha} \) satisfy the integral equations

\[
e^{-i\omega x \cdot \beta} = \int_{\sigma} E(x | \xi) I_{-\beta}(\xi) dS_{\xi}, \quad x \text{ on } \sigma
\]

\[
e^{i\omega x \cdot \alpha} = \int_{\sigma} E(x | \xi) I_{\alpha}(\xi) dS_{\xi}, \quad x \text{ on } \sigma
\]

respectively. Multiplying the first integral equation by \( I_{\alpha}(x) \) and the second by \( I_{-\beta}(x) \) and integrating over \( \sigma \), we find

\[
\int_{\sigma} e^{-i\omega x \cdot \beta} I_{\alpha}(x) dS_x = -4\pi A(\beta, \alpha) = \int_{\sigma} dS_x I_{\alpha}(x) \int_{\sigma} E(x | \xi) I_{-\beta}(\xi) dS_{\xi},
\]

\[
\int_{\sigma} e^{i\omega x \cdot \alpha} I_{-\beta}(x) dS_x = -4\pi A(-\alpha, -\beta) = \int_{\sigma} dS_x I_{-\beta}(x) \int_{\sigma} E(x | \xi) I_{\alpha}(\xi) dS_{\xi}.
\]

Since \( E(x | \xi) = E(\xi | x) \), the double integrals are equal and therefore

\[
A(-\alpha, -\beta) = A(\beta, \alpha), \quad (7.254)
\]

a relation known as the *reciprocity principle*. Thus if we interchange the directions of observation and incidence, the far-field amplitude is unchanged. In other words, the far-field amplitude in the \( \beta \) direction, when the plane wave travels in the \( \alpha \) direction, is equal to the far-field amplitude in the \(-\alpha\) direction when the wave is incident from the \(-\beta\) direction. It is remarkable that the principle holds regardless of the type of obstacle and can even be extended to problems with different boundary conditions (see Exercise 7.47).

**Scattering Cross Section**

Let \( u \) be a solution of the Helmholtz equation with \( \lambda = \omega^2 \) and \( \omega \) real, positive. Then, according to (7.148), the energy flux through a closed surface \( \Sigma \) is given by

\[
P = \frac{\omega}{4i} \int_{\Sigma} \left( \bar{u} \frac{\partial u}{\partial n} - u \frac{\partial \bar{u}}{\partial n} \right) dS = \frac{\omega}{2} \text{Im} \int_{\Sigma} \bar{u} \frac{\partial u}{\partial n} dS. \quad (7.255)
\]

For a plane wave the energy flux per unit area through a plane perpendicular to the direction of propagation is

\[
P_i = \frac{\omega^2}{2}. \quad (7.256)
\]
Now let \( u_s \) be the scattered field when a plane wave is incident on an obstacle. The energy flux through any surface \( \Sigma \) enclosing the obstacle is given by (7.255) with \( u \) replaced by \( u_s \). This flux is independent of the surface \( \Sigma \) (as can be seen from Green's theorem, applied to \( u_s \) and \( \bar{u}_s \)) and can therefore be expressed as

\[
P_s = \frac{\omega}{2} \text{Im} \int_{\sigma} u_s \frac{\partial u_s}{\partial n} \, dS = \frac{\omega}{2} \text{Im} \left[ \lim_{r \to \infty} \int_{\sigma_r} \bar{u}_s \frac{\partial u_s}{\partial n} \, dS \right],
\]

where \( \sigma \) is the boundary of the obstacle and \( \sigma_r \) is a large sphere enclosing the obstacle.

A measure of the effect of the obstacle is the scattering cross section \( D \), which is defined as the ratio

\[
D = \frac{P_s}{P_i}.
\]

Thus

\[
D = \frac{1}{\omega} \text{Im} \int_{\sigma} u_s \frac{\partial u_s}{\partial n} \, dS = \frac{1}{\omega} \text{Im} \left[ \lim_{r \to \infty} \int_{\sigma_r} \bar{u}_s \frac{\partial u_s}{\partial n} \, dS \right],
\]

which can be transformed, in view of (7.252), into

\[
D = \lim_{r \to \infty} \int_{\sigma_r} |u_s|^2 \, dS,
\]

an expression clearly showing the significance of \( D \) as the "energy" radiated by the scattered field.

From the first equality of (7.258), and the fact that the total field \( u \) vanishes on \( \sigma \), we obtain

\[
D = -\frac{1}{\omega} \text{Im} \int_{\sigma} \bar{u}_i \frac{\partial u_s}{\partial n} \, dS = -\frac{1}{\omega} \text{Im} \int_{\sigma} \bar{u}_i \frac{\partial u}{\partial n} \, dS + \frac{1}{\omega} \text{Im} \int_{\sigma} \bar{u}_i \frac{\partial u_i}{\partial n} \, dS.
\]

The last integral vanishes, since \( u_i \) satisfies the homogeneous Helmholtz equation in the interior of \( \sigma \) [see (7.150)]. If the incoming wave propagates in the \( \alpha \) direction, the corresponding scattering cross section is denoted by \( D_\alpha \), and

\[
D_\alpha = -\frac{1}{\omega} \text{Im} \int_{\sigma} e^{-i\alpha \cdot x} \cdot A(\alpha, x) dS.
\]

Comparing with (7.253), we see that

\[
D_\alpha = \frac{4\pi}{\omega} \text{Im} A(\alpha, \alpha),
\]

where \( A(\alpha, \alpha) \) is the forward scattered amplitude in the far zone, since it is observed in the \( \alpha \) direction, which is the direction of propagation of the incident plane wave. It is remarkable that \( D_\alpha \) can be expressed solely in terms of
the value of the scattered field at a single point in space—the forward direction at \( \infty \).

**Low-Frequency Behavior**

With the incident field \( u_i = e^{i \omega x \cdot \alpha} \), the integral equation (7.250) can be solved for small \( \omega \) by a power series expansion in \( \omega \). We set

\[
I_a(x) = \sum_{j=0}^{\infty} A_j(x) \omega^j,
\]

and using the power series for the exponential function, we find that (7.250) becomes, for \( x \) on \( \sigma \),

\[
\sum_{j=0}^{\infty} \frac{(i \omega)^j (x \cdot \alpha)^j}{j!} = \int_{\sigma} \frac{1}{4\pi |x - \xi|} \sum_{j=0}^{\infty} \frac{(i \omega)^j (|x - \xi|)^j}{j!} \cdot \sum_{j=0}^{\infty} A_j(\xi) \omega^j \, dS_\xi.
\]

(7.260)

The zeroth-order terms (that is, the terms in \( \omega^0 \)) yield the integral equation for \( A_0 \):

\[
1 = \int_{\sigma} \frac{1}{4\pi |x - \xi|} A_0(\xi) dS_\xi, \quad x \text{ on } \sigma,
\]

(7.261)

so that \( A_0(x) \) is real.

This is the familiar equation (6.140) for the charge density \( A_0(x) \) of a conductor \( \sigma \) at unit potential. The terms involving the first power of \( \omega \) give the integral equation

\[
i \alpha \cdot x = \int_{\sigma} \frac{i}{4\pi} A_0(\xi) dS_\xi + \int_{\sigma} \frac{1}{4\pi |x - \xi|} A_1(\xi) dS_\xi, \quad x \text{ on } \sigma,
\]

(7.262)

which can be viewed as an integral equation for \( A_1(\xi) \) assuming that \( A_0 \) has been determined from (7.261). Now, by the definition of the capacity \( C \) of a charged conductor, we have

\[
\int_{\sigma} A_0(\xi) dS_\xi = C,
\]

(7.263)

so that (7.262) reduces to

\[
i \left[ \alpha \cdot x - \frac{C}{4\pi} \right] = \int_{\sigma} \frac{1}{4\pi |x - \xi|} A_1(\xi) dS_\xi.
\]

(7.264)

This shows that \( A_1 \) is pure imaginary. From (7.260) one can then find successive integral equations for \( A_2, A_3, \ldots \). One can easily show that \( A_{2j} \) is real and \( A_{2j+1} \) is pure imaginary. By (7.259) we can express the scattering cross section as

\[
D_\alpha = -\frac{1}{2 \omega} \text{Im} \left( \int_{\sigma} \sum_{j=0}^{\infty} \frac{(-i \omega x \cdot \alpha)^j}{j!} \sum_{j=0}^{\infty} A_j(x) \omega^j \, dS \right).
\]
and the first nonvanishing term yields the low-frequency approximation

\[ D_\alpha \sim \int_{\sigma} [iA_1(x) + (x \cdot \omega)A_0(x)] dS. \]  

(7.265)

This integral is easily calculated by multiplying (7.261) by \( A_1 \) and (7.264) by \( A_0 \), subtracting, and integrating over \( \sigma \). This yields

\[ \int_{\sigma} \left[ A_1(x) + i \frac{C}{4\pi} A_0(x) - i\omega \cdot x A_0(x) \right] dS = 0, \]

and, from (7.265), and (7.263),

\[ D_\alpha \sim \frac{C^2}{4\pi}, \quad \omega \text{ small.} \]  

(7.266)

We observe that to this lowest order the scattering cross section is independent of the direction of the incoming plane wave and can be expressed in terms of a purely static quantity—the capacity \( C \).

Additional terms in the power series expansions for \( I_\alpha \) and \( D_\alpha \) can be obtained without great difficulty.

**High-Frequency Behavior**

For large \( \omega \) one might hope for a series expansion for \( I_\alpha \) and \( D_\alpha \) in inverse powers of \( \omega \). Such series exist but involve nonintegral powers and are asymptotic instead of convergent. Considerable attention has been given to this difficult problem in recent years, but only some simple aspects will be treated here. By examining the equation \( \nabla^2 u_s + \omega^2 u_s = 0 \), we can see the nature of the difference for small and large \( \omega \). If \( \omega \) is small, the term \( \omega^2 u_s \) may be regarded as a perturbation on the potential equation which is of the same order and type as the Helmholtz equation; on the other hand, if \( \omega \) is large, one has

\[ u_s + \frac{1}{\omega^2} \nabla^2 u_s = 0 \]

and the nature of the problem is radically changed as \( \omega \to \infty \) (a singular perturbation).

We begin by examining the reflection of a monochromatic plane wave by a plane mirror. Let the incoming wave be

\[ u_i = e^{i\omega x \cdot \alpha}, \]

which travels in the \( \alpha \) direction (see Figure 7.14). The boundary condition on the mirror \( \sigma \) is that the total field \( u \) must vanish. The plane containing \( \alpha \) and the normal \( n \) to the mirror has been taken as the plane of the paper. The total field \( u \) satisfies the homogeneous equation to the left of \( \sigma \) and vanishes on \( \sigma \). We write

\[ u = u_i + u_r = e^{i\omega x \cdot \alpha} + u_r; \quad u = 0, \quad x \text{ on } \sigma, \]

wh
where $u_r$ is to be found. Unfortunately this problem does not have a unique solution unless an appropriate condition is imposed on $u_r$ in the far field. Rather than trying to characterize this condition, we think of our problem as the limit of a problem in which the incident field is due to a point source at $P$ removed to infinity along the $-\alpha$ direction (see Figure 7.15). The point-source problem is easily solved by placing an image source at $P^*$; in the limit
as $P$ (and hence $P^*$) are removed to infinity the incident field becomes proportional to $e^{i\omega x \cdot \alpha}$ and the reflected field to $e^{i\omega x \cdot \alpha^*}$, where $\alpha^*$ is shown in Figure 7.14, and

$$\alpha^* = \alpha - 2(\alpha \cdot n)n$$

is a unit vector.

This leads us to surmise that

$$u = e^{i\omega x \cdot \alpha} + A e^{i\omega x \cdot \alpha^*},$$

where $A$ must be calculated to satisfy the boundary condition

$$0 = e^{i\omega x \cdot \alpha} + A e^{i\omega x \cdot \alpha} e^{-2i\omega (\alpha \cdot n)(x \cdot n)}, \quad x \text{ on } \sigma.$$ 

If $x_0$ denotes a fixed point on the screen, then the equation of the screen is $(x - x_0) \cdot n = 0$. Hence

$$A = -e^{2i\omega (\alpha \cdot n)(x_0 \cdot n)}.$$ 

We shall be interested in the current $\partial u/\partial n$ on the screen; we have

$$\frac{\partial u}{\partial n} = n \cdot \text{grad } u = i\omega e^{i\omega x \cdot \alpha}(n \cdot \alpha) + i\omega A e^{i\omega x \cdot \alpha^*}(n \cdot \alpha^*).$$

Since $n \cdot \alpha^* = -n \cdot \alpha$, we find

$$\left(\frac{\partial u}{\partial n}\right)_\sigma = 2i\omega (n \cdot \alpha) e^{i\omega x \cdot \alpha}, \quad x \text{ on } \sigma.$$ 

Next consider the high-frequency scattering of a plane wave $e^{i\omega x \cdot \alpha}$ by a smooth, convex body as in Figure 7.16. Owing to the rapid oscillations in the incoming field, the obstacle appears locally (say, at $P$) to be a plane. Thus the plane wave striking the obstacle at $P$ will be reflected as shown. Since the body is convex, the reflected wave does not strike the body again. Therefore, on the illuminated side we may take the current to be that due to local reflection of a plane wave; that is,

$$I_{\text{ill}} = 2i\omega (n \cdot \alpha) e^{i\omega x \cdot \alpha},$$
where \( n \) is the local normal. On the dark side we take \( I = 0 \). From (7.259) we may calculate the scattering cross section

\[
D_\alpha = -\frac{1}{\omega} \text{Im} \int_{\sigma_{\text{III}}} 2i\omega(n \cdot \alpha) dS = -2 \int_{\sigma_{\text{III}}} (n \cdot \alpha) dS = 2A_p, \tag{7.267}
\]

where \( A_p \) is the projected cross-sectional area of the obstacle on a plane perpendicular to the \( \alpha \) direction. Although (7.267) is the correct geometric optics cross section corresponding to \( \omega = \infty \), the method gives no hint how to obtain corrections for large finite \( \omega \).

**Stationary Principle for the Cross Section**

According to (7.253), the forward scattered amplitude \( A(\alpha, \alpha) \) is given by

\[
-4\pi A(\alpha, \alpha) = \int_{\sigma} e^{-i\omega x \cdot \alpha} I_\alpha(x) dS_x. \tag{7.268}
\]

Thus \( A(\alpha, \alpha) \) is a linear functional of \( I_\alpha \), the solution of the integral equation

\[
e^{i\omega x \cdot \alpha} = \int_{\sigma} E(x \mid \xi) I_\alpha(\xi) dS_\xi, \quad x \text{ on } \sigma, \tag{7.269}
\]

where

\[
E(x \mid \xi) = E(\xi \mid x) = \frac{e^{i\omega |x - \xi|}}{4\pi |x - \xi|}.
\]

Since the scattering cross section \( D_\alpha \) is proportional to the imaginary part of \( A(\alpha, \alpha) \), we may focus our attention on \( A(\alpha, \alpha) \). If the integral equation could be solved exactly, (7.253) would provide us with the exact value of \( A(\alpha, \alpha) \). Unfortunately, it is usually difficult to solve the integral equation exactly and we would therefore like to estimate the forward scattered amplitude in terms of an approximate solution of the integral equation. To achieve greater accuracy in calculating \( A(\alpha, \alpha) \), we look for a formula which is stationary about the solution of the integral equation and thus relatively insensitive to small errors in \( I_\alpha \). The general formulation of stationary principles is discussed in Chapter 8; here we restrict ourselves to the case of integral equations of the first kind.

Consider then the problem of estimating the functional

\[
\gamma = \int_{\Omega} b(\xi) f(\xi) d\xi, \tag{7.270}
\]

where \( f \) satisfies the integral equation

\[
a(x) = \int_{\Omega} k(x, \xi) f(\xi) d\xi, \quad x \text{ in } \Omega. \tag{7.271}
\]
Here \( a, b, \) and \( k \) are given, complex-valued functions, and \( k(x, \xi) = k(\xi, x) \). We introduce the auxiliary integral equation

\[
b(x) = \int_{\mathbb{R}} k(x, \xi) g(\xi) d\xi, \quad x \text{ in } \mathbb{R}. \tag{7.272}
\]

where \( g \), as well as \( f \), is unknown. Multiplying (7.271) by \( g \) and (7.272) by \( f \), subtracting, and integrating over \( \mathbb{R} \), we obtain, in view of the relation \( k(x, \xi) = k(\xi, x) \),

\[
\gamma = \int_{\mathbb{R}} b(x)f(x)dx = \int_{\mathbb{R}} a(x)g(x)dx = \int_{\mathbb{R}} \int_{\mathbb{R}} k(x, \xi)f(x)g(\xi)dx\,d\xi. \tag{7.273}
\]

The particular combination

\[
\int_{\mathbb{R}} b(x)f(x)dx + \int_{\mathbb{R}} a(x)g(x)dx - \int_{\mathbb{R}} \int_{\mathbb{R}} k(x, \xi)f(x)g(\xi)dx\,d\xi \tag{7.274}
\]

is therefore also equal to \( \gamma \). The advantage of this expression over the apparently simpler one (7.270) is that it is stationary with respect to small variations in \( f \) and \( g \).

To simplify the notation we let

\[
\langle u, v \rangle = \int_{\mathbb{R}} uv\,dx
\]

\[
Ku = \int_{\mathbb{R}} k(x, \xi)u(\xi)d\xi.
\]

We observe that \( \langle Ku, v \rangle = \langle u, Kv \rangle \), for any \( u, v \).

**Theorem.** Let

\[
F(u, v) = \langle b, u \rangle + \langle a, v \rangle - \langle Ku, v \rangle. \tag{7.275}
\]

Then \( F \) is stationary about \( u = f, v = g \), where \( f \) and \( g \) are the respective solutions of (7.271) and (7.272). Moreover, the stationary value of \( F \) is \( \gamma \).

**Proof.** We want to show that

\[
\Delta F = F(f + \delta f, g + \delta g) - F(f, g)
\]

is of second order in the variations \( \delta f, \delta g \). We find

\[
\Delta F = \langle b, \delta f \rangle + \langle a, \delta g \rangle - \langle Kf, \delta g \rangle - \langle K\delta f, g \rangle - \langle K\delta f, \delta g \rangle
\]

\[= \langle b - Kg, \delta f \rangle + \langle a - Kf, \delta g \rangle - \langle K\delta f, \delta g \rangle.\]

By assumption, \( b = Kg, a = Kf \), so that \( \Delta F \) is of the second order, as was to be shown.

To use (7.275) to estimate \( \gamma \), we must substitute for \( u \) an approximation to the solution \( f \) of (7.271) and for \( v \) an approximation to the solution \( g \) of (7.272).
COROLLARY (SCHWINGER-LEVINE PRINCIPLE). Let

\[ R(u, v) = \frac{\langle b, u \rangle \langle a, v \rangle}{\langle Ku, v \rangle}. \quad (7.276) \]

Then \( R \) is stationary about \( u = cf, v = dg \), where \( c \) and \( d \) are arbitrary constants. Moreover \( R(cf, dg) = \gamma \) for all nonzero \( c \) and \( d \).

The advantage of (7.276) is that we only have to guess the form of the solutions of the integral equations, not their size.

We now apply (7.276) to the calculation of the forward scattered amplitude (7.268). We have

\[
\begin{align*}
f &= I_\alpha, & k(x, \xi) &= \frac{e^{i\alpha|x-\xi|}}{4\pi|x-\xi|}, & R &= \sigma, \\
a(x) &= e^{i\alpha x \cdot a}, & b(x) &= e^{-i\alpha x \cdot a}, & \gamma &= A(\alpha, \alpha).
\end{align*}
\]

These identifications imply that \( g \) is the solution of the integral equation (7.269) with inhomogeneous term \( e^{-i\alpha x \cdot a} \). Thus \( g \) must be set equal to \( I_{-a}(x) \), the current induced by a wave traveling in the \(-a\) direction. Then we have

\[
-4\pi A(\alpha, \alpha) = \text{stationary value} \frac{\int \int e^{-i\alpha x \cdot a} dS \int \int e^{i\alpha x \cdot a} dS}{\int \int dS_x dS_\xi (e^{i\alpha|x-\xi|/4\pi|x-\xi|}u(x)v(\xi))},
\]

(7.277)

where for \( u \) we must substitute an approximation to \( I_\alpha \) and for \( v \) an approximation to \( I_{-a} \).

7.16 WIENER-HOPF METHOD

In this section we consider integral equations of the type

\[
\int_a^b k(x - \xi)u(\xi)d\xi = \mu u(x) + f(x), \quad a < x < b
\]

(7.278)

where \( \mu, f, \) and \( k \) are given and we wish to find \( u(x), \ a < x < b \). In Chapter 3, we considered a more general problem, where the kernel was an arbitrary function of \( x \) and \( \xi \), whereas here the kernel \( k(x - \xi) \), a difference kernel, is generated from a function \( k(x) \) of a single variable by substituting \( x - \xi \) for \( x \). Of course, the methods previously described apply here also at least in the case of a symmetric kernel \( k(x - \xi) = \overline{k(\xi - x)} \), but, unfortunately, these methods involve eigenfunction expansions and their value is theoretical rather than practical. Here we shall describe an entirely different approach, based on Fourier transforms, for solving (7.278) in the case where one or both of the limits of integration are infinite.
**Interval** $(-\infty, \infty)$

First we investigate the relatively simple case where $a = -\infty$ and $b = +\infty$. Then equation (7.278) becomes

$$
\int_{-\infty}^{\infty} k(x - \xi)u(\xi)d\xi = \mu u(x) + f(x), \quad -\infty < x < \infty.
$$

(7.279)

Taking the Fourier transform of this equation, we obtain, formally, by using the convolution theorem,

$$
k^\wedge(\omega)u^\wedge(\omega) = \mu u^\wedge(\omega) + f^\wedge(\omega),
$$

$$
u^\wedge(\omega) = \frac{f^\wedge(\omega)}{k^\wedge(\omega) - \mu}.
$$

(7.280)

Since the right side is known, it remains only to invert. Precise conditions for the validity of the method can be given, but we shall not do so here. It is worth observing, though, that the homogeneous equation ($f = 0$) may have nontrivial solutions. In fact, the transformed equation in this case is

$$
[k^\wedge(\omega) - \mu]u^\wedge(\omega) = 0,
$$

(7.281)

and if $k^\wedge(\omega) - \mu$ has a simple zero at $\omega = \omega_0$, we may choose

$$
u^\wedge(\omega) = c\delta(\omega - \omega_0),
$$

(7.282)

since it was shown in Volume I that, in the distributional sense,

$$(\omega - \omega_0)\delta(\omega - \omega_0) = 0.
$$

On inverting (7.282) we obtain

$$
u(x) = \text{constant } e^{-i\omega_0 x}.
$$

If $k^\wedge(\omega) - \mu$ has a zero of order $p$ at $\omega = \omega_0$, then

$$
u^\wedge(\omega) = \sum_{j=0}^{p-1} c_j\delta^{(j)}(\omega - \omega_0)
$$

is a solution of (7.281), leading to

$$
u(x) = \sum_{j=0}^{p-1} c_j x^j e^{-i\omega_0 x}.
$$

If the homogeneous equation has only the trivial solution, that is, if $k^\wedge(\omega) - \mu$ has no zeros, then (7.280) will, under fairly mild additional conditions, provide the one and only solution of (7.279).

**Example**

Let $k(x) = e^{-|x|}$, $k^\wedge(\omega) = 2/1 + \omega^2$. First we look at the homogeneous equation

$$
\int_{-\infty}^{\infty} e^{-|x| - i\xi}u(\xi)d\xi = \mu u(x), \quad -\infty < x < \infty,
$$

(7.283)
for real $\mu$. The transformed equation is
\[
\left[ \frac{2}{1 + \omega^2} - \mu \right] u^\wedge(\omega) = 0.
\]
If $\mu = 1$ the simple zeros of the term in brackets are at $\omega = \pm 1$. Therefore
\[
Ae^{ix} + Be^{-ix}
\]
is the solution of the integral equation, as is easily verified. If $\mu = 2$, the term in brackets has a double zero at $\omega = 0$ and
\[
A + Bx
\]
is the solution of the homogeneous equation. If $\mu < 0$, the bracketed term has zeros at
\[
\omega = \pm ir,
\]
where $r$ is real and greater than 1. This implies
\[
u = Ae^{rx} + Be^{-rx},
\]
which makes the integral in (7.283) diverge and is therefore not a solution.

Next consider the inhomogeneous equation
\[
\int_{-\infty}^{\infty} e^{-|x-\xi|^\frac{1}{2}} u(\xi)d\xi = -\frac{1}{4}u(x) + e^{-|x|}, \quad -\infty < x < \infty.
\]
Equation (7.280) yields
\[
\frac{u^\wedge(\omega)}{9 + \omega^2} = \frac{8}{9 + \omega^2},
\]
\[
u(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x} \frac{8}{9 + \omega^2} d\omega.
\]
Evaluating the integral for $x > 0$ by using a semicircular contour in the lower half of the $\omega$ plane, we obtain
\[
u(x) = \frac{4}{3}e^{-3x}, \quad x > 0.
\]
For negative $x$, we use a contour in the upper half-plane to find $u = -\frac{4}{3}e^{3x}$, so that
\[
u = \frac{4}{3}e^{-3|\cdot|}, \quad -\infty < x < \infty.
\]

**Interval** $(0, \infty)$

The integral equation (7.278) becomes
\[
\int_{0}^{\infty} k(x - \xi)u(\xi)d\xi = \mu u(x) + f(x), \quad 0 < x < \infty.
\]
This is the problem to which the Wiener-Hopf procedure is applicable. Its distinguishing features are the difference kernel and the semiinfinite integral. To calculate the integral on the left for positive \( x \), we need \( k \) for values of the argument between \(-\infty\) and \( \infty \). Thus (7.284) is a legitimate integral equation only if \( k(x) \) is given for \(-\infty < x < \infty\), and we shall, of course, assume that this is the case.

To apply Fourier transforms we must extend the integral equation to the infinite interval \(-\infty < x < \infty\). Since \( k \) is known for all values of its argument, the left side of (7.284) defines a function \( g(x) \) for \( x < 0 \). Of course, \( g(x) \) is unknown until the equation (7.284) has been solved for \( u(x) \). In any event,

\[
\int_0^\infty k(x - \xi)u(\xi)d\xi = \begin{cases} 
\mu u(x) + f(x), & 0 < x < \infty \\
g(x), & -\infty < x < 0.
\end{cases}
\]

By extending the definitions of \( u, f, \) and \( g \) as follows:

\[
u = 0, \quad x < 0; \quad f = 0, \quad x < 0; \quad g = 0, \quad x > 0,
\]

we can rewrite (7.284)

\[
\int_{-\infty}^\infty k(x - \xi)u_+(\xi)d\xi = \mu u_+(x) + f_+(x) + g_-(x), \quad -\infty < x < \infty,
\]

(7.285)

where a subscript indicates the half-line on which the function is nonvanishing. Functions such as \( f_+(x) \) and \( u_+(x) \) which vanish for negative \( x \) are said to be right-sided functions, whereas \( g_-(x) \) is a left-sided function. We now list the assumptions under which we will attempt to solve (7.285):

1. \( k(x) = O(e^{-c|x|}) \) as \( |x| \to \infty \), \( c > 0 \).
2. \( f(x) = O(e^{d'x}) \) as \( x \to +\infty \), \( d' < c \).

The integral on the left side will exist if \( u \) is of exponential order at infinity with exponent smaller than \( c \). Thus we shall look for a solution \( u \) satisfying

3. \( u(x) = O(e^{d''x}) \) as \( x \to +\infty \), \( d'' < c \).

From (1) and (3), it then follows that

4. \( g(x) = O(e^{-c|x|}) \) as \( x \to -\infty \).

Taking the Fourier transform of (7.285), we find

\[
k^\wedge(\omega)u^+_\wedge(\omega) - \mu u^+_\wedge(\omega) = f^+_\wedge(\omega) + g^\wedge(\omega),
\]

(7.286)

where, according to our discussion in Section 5.6, the transforms are defined and analytic in the following regions:

\[
k^\wedge(\omega) \quad \text{in the strip} \quad -c < \text{Im} \ \omega < c;
\]

\[
u^+_\wedge(\omega) \quad \text{in the upper half-plane} \quad \text{Im} \ \omega > d'';
\]

\[
f^+_\wedge(\omega) \quad \text{in the upper half-plane} \quad \text{Im} \ \omega > d';
\]

\[
g^\wedge(\omega) \quad \text{in the lower half-plane} \quad \text{Im} \ \omega < c.
\]
If we let \( d = \max (d', d'') \), then all functions are analytic in the strip
\[
d < \text{Im } \omega < c,
\]
as in Figure 7.17, and therefore (7.286) holds in that strip.

The notation \( u^\wedge(\omega) \) deserves a fuller explanation. It does not mean that the function \( u^\wedge(\omega) \) vanishes for negative \( \omega \) but rather that it is the transform of a right-sided function \( u_+(x) \). It is a convenient coincidence that \( u^\wedge(\omega) \), being the transform of a right-sided function, is automatically analytic in an upper half-plane.

![Diagram](image)

**Figure 7.17**

Equation (7.286) has two unknowns, \( u^\wedge \) and \( g^\wedge \). Remarkably enough, a function-theoretic method (the Wiener-Hopf method) enables us to find both \( u^\wedge \) and \( g^\wedge \). The idea is to rewrite (7.286) by suitable manipulations in the form
\[
H(\omega) = I(\omega), \quad d < \text{Im } \omega < c, \tag{7.287}
\]
where \( H(\omega) \) is analytic in the upper half-plane \( \text{Im } \omega > d \) and \( I(\omega) \) is analytic in the lower half-plane \( \text{Im } \omega < c \). Here \( H(\omega) \) contains \( u^\wedge \) and known functions whereas \( I(\omega) \) contains \( g^\wedge \) and known functions. Since \( H \) and \( I \) are analytic functions in overlapping regions which together cover the whole \( \omega \) plane, they must be analytic continuations of each other. Hence \( H \) and \( I \) must equal the same entire function \( E(\omega) \), an entire function being one which is analytic for all finite values of \( \omega \). It remains only to find \( E(\omega) \). This is accomplished by estimating \( H \) and \( I \) at infinity in their respective half-planes of definition. In particular, if \( H \) approaches 0 as \( \text{Im } \omega \to +\infty \) and \( I \) approaches 0 as \( \text{Im } \omega \to -\infty \), then, by Liouville's theorem, \( E \) is the function identically 0. It should be noted that (7.286) can lead to many different equations of the form
(7.287)—for instance, multiplication of (7.287) by an entire function such as $e^{i\omega}$ or $\omega^n$ gives another equation of the same form—and it may require skill to obtain (7.287) in a form where the entire function $E(\omega)$ can be calculated.

To avoid the possibility of existence of solutions of the homogeneous equation ($f = 0$) in (7.284), we assume that $k^+(\omega) - \mu$ has no zeros in $-c < \text{Im}\ \omega < c$.

Proceeding from (7.286), we first split $k^+(\omega) - \mu$ as a quotient

$$\frac{k^+(\omega)}{k^-(\omega)},$$

where $k^+$ is analytic in $\text{Im}\ \omega > d$, $k^-$ in $\text{Im}\ \omega < c$. Note that when dealing with functions of $\omega$, the subscripts refer only to regions of analyticity. Here of course, we do not know that $k^+$ and $k^-$ are Fourier transforms. Then (7.286) becomes

$$k^+_+ u^+_+ = k^-_+ g^+_+ + k^-_- f^+_+.$$  \hspace{1cm} (7.288)

Next we split $k^-_- f^+_+$, which is analytic in $d < \text{Im}\ \omega < c$, into a sum

$$h^+_+ + h^-_-,$$

where $h^+_+$ is analytic in $\text{Im}\ \omega > d$ and $h^-_-$ in $\text{Im}\ \omega < c$. This reduces (7.288) to

$$k^+_+ u^+_+ - h^+_+ = k^-_+ g^+_+ + h^-_-,$$  \hspace{1cm} (7.289)

which is of the form (7.287).

The two splittings required can sometimes be performed by inspection, but a systematic procedure based on Cauchy's integral theorem is also available.

**Sum Splitting**

Let $a(\omega)$ be analytic in $d < \text{Im}\ \omega < c$ and let $a(\omega) \to 0$ as $|\text{Re}\ \omega| \to \infty$ within the strip of analyticity. Then, we can decompose $a(\omega)$ unambiguously as the sum

$$a(\omega) = a^+_+(\omega) + a^-_-(\omega),$$  \hspace{1cm} (7.290)

where $a^+_+$ and $a^-_-$ are analytic in $\text{Im}\ \omega > d$ and $\text{Im}\ \omega < c$, respectively, and $a^+_+(\omega) \to 0$ as $\text{Im}\ \omega \to \infty$, $a^-_-(\omega) \to 0$ as $\text{Im}\ \omega \to -\infty$.

The decomposition (7.290) follows from an application of Cauchy's integral theorem to the rectangle of Figure 7.18. Then, if $\omega$ is within the rectangle,

$$2\pi i a(\omega) = \int_{c_1}^{\alpha - \omega} \frac{a(\alpha)}{\alpha - \omega} \, d\alpha + \int_{c_2}^{\alpha - \omega} \frac{a(\alpha)}{\alpha - \omega} \, d\alpha + \int_{c_2 + c_4}^{\alpha - \omega} \frac{a(\alpha)}{\alpha - \omega} \, d\alpha.$$  

Since $a$ approaches 0 as $|\text{Re}\ \alpha| \to \infty$, the integrals on $C_3$ and $C_4$ tend to 0 as the width of the rectangle tends to infinity. Therefore,

$$2\pi i a(\omega) = \int_{-\infty + it}^{+\infty + it} \frac{a(\alpha)}{\alpha - \omega} \, d\alpha - \int_{-\infty + it}^{+\infty + it} \frac{a(\alpha)}{\alpha - \omega} \, d\alpha,$$  \hspace{1cm} (7.291)
where $\tau > d$ and $\sigma < c$. In the first integral we can take $\tau$ as close to $d$ as we please and the function of $\omega$ defined by the integral is clearly analytic in the half-plane $\text{Im} \, \omega > d$; moreover, the integral approaches $0$ as $\text{Im} \, \omega \to \infty$. A similar analysis shows that the second integral is analytic for $\text{Im} \, \omega < c$ and tends to $0$ as $\text{Im} \, \omega \to -\infty$.

As an example, let $a(\omega) = 1/1 + \omega^2$, which is analytic in the strip $-1 < \text{Im} \, \omega < 1$. Then according to (7.290) and (7.291), we can write

$$a(\omega) = a_+(\omega) + a_-(\omega), \quad \tau < \text{Im} \, \omega < \sigma,$$

where

$$a_+(\omega) = \frac{1}{2\pi i} \int_{-\infty + i\tau}^{+\infty + i\tau} \frac{a(\alpha)}{\alpha - \omega} \, d\alpha, \quad \tau > -1, \quad \text{Im} \, \omega > \tau,$$

$$a_-(\omega) = -\frac{1}{2\pi i} \int_{-\infty + i\sigma}^{+\infty + i\sigma} \frac{a(\alpha)}{\alpha - \omega} \, d\alpha, \quad \sigma < 1, \quad \text{Im} \, \omega < \sigma.$$

To calculate $a_+$ we can close the contour either in the upper or lower half-plane; this latter calculation is a little simpler, since it involves only the residue at $\alpha = -i$. We obtain

$$a_+(\omega) = -\text{res} \left( \frac{1}{(1 + \alpha^2)(\alpha - \omega)} \right) \quad \text{at} \ \alpha = -i,$$

or

$$a_+(\omega) = \frac{-1}{-2i(-i - \omega)} = \frac{i}{2(\omega + i)}.$$

Similarly,

$$a_-(\omega) = -\text{res} \left( \frac{1}{(1 + \alpha^2)(\alpha - \omega)} \right) = -\frac{1}{2i(i - \omega)} = \frac{-i}{2(\omega - i)}.$$
We note that \( a_+ (\omega) \) is analytic not only for \( \text{Im} \ \omega > \tau \) but actually for \( \text{Im} \ \omega > -1 \), and \( a_- (\omega) \) for \( \text{Im} \ \omega < 1 \). Adding, we obtain
\[
\frac{i}{2} \left[ \frac{1}{\omega + i} - \frac{1}{\omega - i} \right] = \frac{1}{\omega^2 + 1}.
\]

**Quotient Splitting**

Let \( b(\omega) \) be analytic in the strip \( d < \text{Im} \ \omega < c \) and let \( a(\omega) = \log b(\omega) \) satisfy the hypothesis for the sum splitting discussed above. In particular, this implies that \( b(\omega) \) is nonvanishing in the strip and that \( b(\omega) \to 1 \) as \( |\text{Re} \ \omega| \to \infty \) within the strip. Then we have
\[
a(\omega) = \log b(\omega) = a_+ (\omega) + a_- (\omega).
\]
Setting
\[
b_+ (\omega) = \exp [a_+ (\omega)], \quad b_- (\omega) = \exp [-a_- (\omega)],
\]
we have
\[
b_+ (\omega)/b_- (\omega) = \exp [a_+ (\omega) + a_- (\omega)] = b(\omega),
\]
and the required decomposition has been found.

To find \( a_+ \) and \( a_- \) we may use (7.291) with \( a(x) = \log b(x) \), but the presence of the logarithms often makes the evaluation of the integrals quite complicated. In the examples the decomposition will be achieved by inspection.

**Examples of the Wiener-Hopf Method**

The first example does not arise from any physical problem but illustrates the decompositions required in the Wiener-Hopf method. Consider the integral equation
\[
\int_0^\infty e^{-|x-\xi|} u(\xi) d\xi = -\frac{1}{4} u(x) + 1, \quad 0 < x < \infty.
\]
(7.292)

Since
\[
u(x) = 4 - 4 \int_0^\infty e^{-|x-\xi|} u(\xi) d\xi,
\]
it is easily seen that \( u(x) \) is bounded as \( x \to + \infty \). Therefore, \( u_+ (\omega) \) will be analytic in \( \text{Im} \ \omega > 0 \). The transform of
\[
f_+(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0, \end{cases}
\]
is
\[
f_+ (\omega) = \frac{i}{\omega}, \quad \text{Im} \ \omega > 0.
\]
The kernel \( k(x) = e^{-|x|} \) has the transform
\[
k^\wedge(\omega) = \frac{2}{1 + \omega^2},
\]
which is defined and analytic in the strip
\[-1 < \text{Im } \omega < 1.\]

The function
\[
g_-(x) = \begin{cases} 
\int_0^\infty e^{-|x-\xi|} u(\xi) d\xi, & x < 0, \\
0, & x > 0,
\end{cases}
\]
has an unknown transform \( g_-(\omega) \) which is analytic for \( \text{Im } \omega < 1 \). In our special case, (7.286) takes the form
\[
\frac{9 + \omega^2}{4(1 + \omega^2)} u_+^\wedge(\omega) = \frac{i}{\omega} + g_+^\wedge(\omega), \quad 0 < \text{Im } \omega < 1, \quad (7.293)
\]
where the regions of analyticity of the various terms are shown in Figure 7.19. Our first step is to split \((9 + \omega^2)/4(1 + \omega^2)\) into a quotient of functions analytic

![Figure 7.19](http://www.mathschoolinternational.com)
in upper and lower half-planes, respectively. This can be done by inspection. Indeed,

\[
\frac{9 + \omega^2}{4(1 + \omega^2)} = \frac{k_+(\omega)}{k_-(\omega)},
\]

where

\[
k_+ = \frac{\omega + 3i}{4(\omega + i)}
\]
is analytic in \(\text{Im } \omega > -1\), and

\[
k_- = \frac{\omega - i}{\omega - 3i}
\]
is analytic in \(\text{Im } \omega < 1\).

Note that the decomposition is not unique, since we can multiply \(k_+\) and \(k_-\) by the same arbitrary entire function. The present decomposition has the advantages of simplicity and of predictable behavior of the factors at \(\text{Im } \omega = +\infty\) and \(\text{Im } \omega = -\infty\). Thus (7.293) becomes

\[
\frac{\omega + 3i}{4(\omega + i)} u^\wedge_+(\omega) = \frac{i(\omega - i)}{(\omega - 3i)\omega} + \frac{\omega - i}{\omega - 3i} g^\wedge_-(\omega). \tag{7.294}
\]

The left side is analytic for \(\text{Im } \omega > 0\); the last term for \(\text{Im } \omega < 1\). The term

\[
\frac{i(\omega - i)}{(\omega - 3i)\omega}
\]
is analytic in the strip \(0 < \text{Im } \omega < 1\), and must be split as a sum of terms, one analytic for \(\text{Im } \omega > 0\), the other for \(\text{Im } \omega < 1\). Again, by inspection, using partial fractions,

\[
\frac{i(\omega - i)}{\omega(\omega - 3i)} = \frac{i}{3\omega} + \frac{2i}{3(\omega - 3i)},
\]

where \(i/3\omega\) is analytic for \(\text{Im } \omega > 0\), the remaining term being analytic for \(\text{Im } \omega < 3\), hence surely for \(\text{Im } \omega < 1\). Therefore (7.294) becomes

\[
\frac{\omega + 3i}{4(\omega + i)} u^\wedge_+(\omega) - \frac{i}{3\omega} = \frac{2i}{3(\omega - 3i)} + \frac{\omega - i}{\omega - 3i} g^\wedge_-(\omega), \quad 0 < \text{Im } \omega < 1.
\]

The left side is analytic for \(\text{Im } \omega > 0\), the right side for \(\text{Im } \omega < 1\), so that both sides are equal to the same entire function \(E(\omega)\). As \(\text{Im } \omega \to +\infty\), the left side goes to 0, and as \(\text{Im } \omega \to -\infty\), the right side tends to 0; therefore, \(E = 0\). Hence

\[
u^\wedge_+(\omega) = \frac{4i}{3} \frac{\omega + i}{\omega(\omega + 3i)},
\]
and, for \( x > 0 \),
\[
    u(x) = \frac{4i}{(3)(2\pi)} \int_{ia-\infty}^{ia+\infty} e^{-i\omega x} \frac{\omega + i}{\omega(\omega + 3i)} d\omega, \quad a > 0.
\]

We evaluate the integral by choosing the contour in the lower half-plane, taking advantage of the fact that \( e^{-i\omega x} \) is small on a lower semicircle when \( x > 0 \). By residues, we find
\[
    u(x) = \frac{4}{3} \left[ \frac{1}{3} + \frac{2}{3} e^{-3x} \right] = \frac{4}{9} [1 + 2e^{-3x}].
\]

It is easy to verify that this function \( u(x) \) actually satisfies (7.292). Equation (7.292) can also be solved by reduction to a differential equation of the second order (see Exercise 7.55).

Our second example is one considered previously—the half-plane excited by a line source. We take the line source to lie in the extension of the half-plane, the trace of the source in the \( xy \) plane being at \((-a, 0)\), where \( a > 0 \). The field \( w(x, y) \) satisfies (7.206), which, in Cartesian coordinates, becomes
\[
    -\frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} + k^2 w = \delta(x + a)\delta(y), \quad 0 < x < \infty.
\]

Here we have taken \( \lambda = -k^2, \ k > 0 \), and we shall show later how to pass to arbitrary \( \lambda \). The advantage of our approach is that we are then dealing with fields which are exponentially decaying at \( \infty \). The total current on the screen is defined as
\[
    I(x) = \frac{\partial w}{\partial y} (x, 0+) - \frac{\partial w}{\partial y} (x, 0-), \quad x > 0. \quad (7.296)
\]

As was shown in (7.250), the current satisfies the integral equation
\[
    E(x, 0+ - a, 0) = \int_0^\infty E(x, 0|\xi, 0) I(\xi) d\xi, \quad 0 < x < \infty,
\]
where \( E \) is the free-space fundamental solution
\[
    E(x, y|\xi, \eta) = \frac{1}{2\pi} K_0[k((x - \xi)^2 + (y - \eta)^2)^{1/2}],
\]
\( K_0 \) being the Macdonald function. Thus the integral equation for \( I \) is
\[
    \frac{1}{2\pi} K_0[k(x + a)] = \int_0^\infty \frac{1}{2\pi} K_0(k|x - \xi|) I(\xi) d\xi, \quad x > 0. \quad (7.297)
\]
The kernel is \(0(e^{-k|x|})\) at \(x = +\infty\), as can be inferred from the known asymptotic behavior of \(K_0\) at \(\infty\). The Fourier transform of the kernel is

\[
\left[ \frac{1}{2\pi} K_0(k|x|) \right]^\wedge = \int_{-\infty}^{\infty} e^{i\omega x} \frac{1}{2\pi} K_0(k|x|) dx = \frac{1}{2\sqrt{\omega^2 + k^2}},
\]

(7.298)

where \(\sqrt{\omega^2 + k^2}\) must be chosen as the square root which is analytic \(-k < \text{Im} \omega < k\) and such that, for \(\omega = 0\), \(\sqrt{\omega^2 + k^2} = k\). The result (7.298) is easily obtained by taking the Fourier transform on \(x\) of the differential equation satisfied by the free-space fundamental solution.

The current \(I(x)\) on the screen can be expected to be bounded at \(x = +\infty\) (in fact, it will be exponentially decaying, but we do not need to assume this); therefore if we define

\[
I_+(x) = \begin{cases} I(x), & x > 0, \\ 0, & x < 0, \end{cases}
\]

which, incidentally, is consistent with (7.296), then \(I_+^\wedge(\omega)\) will be analytic in \(\text{Im} \omega > 0\).

We can then rewrite (7.297) as

\[
\int_{-\infty}^{\infty} \frac{1}{2\pi} K_0(k|x - \xi|) I_+(\xi)d\xi = \frac{1}{2\pi} K_0[k(x + a)] - g_-(x), \quad -\infty < x < \infty,
\]

(7.299)

where \(g_-\) is defined in a slightly different way than in (7.285); here we take

\[
g_-(x) = \begin{cases} 0, & x > 0; \\ \int_{0}^{\infty} \frac{1}{2\pi} K_0(k|x - \xi|) I(\xi)d\xi + \frac{1}{2\pi} K_0[k(x + a)], & x < 0. \end{cases}
\]

(7.300)

With this definition \(g_-(x)\) has a simple physical interpretation: It is the total field \(w\) on the line \(y = 0\). The mathematical advantage of this formulation is that the two-sided transform of

\[
\frac{1}{2\pi} K_0[k(x + a)]
\]

is more easily calculated than its one-sided transform. In fact, we have

\[
\left\{ \frac{1}{2\pi} K_0[k(x + a)] \right\}^\wedge = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega x} K_0[k(x + a)] dx
\]

\[
= \frac{e^{-i\omega a}}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} K_0(kt) dt,
\]

and, since \(K_0\) is an even function, we can use (7.298) to find

\[
\left\{ \frac{1}{2\pi} K_0[k(x + a)] \right\}^\wedge = \frac{e^{-i\omega a}}{2\sqrt{\omega^2 + k^2}}, \quad -k < \text{Im} \omega < k.
\]


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The function \( g_+(x) \), defined by (7.300), is easily seen to be \( 0(e^{-k|x|}) \) as \( x \to -\infty \), and, therefore, \( g_+^\wedge(\omega) \) is analytic for \( \text{Im} \ \omega < k \). Thus by taking the Fourier transform of (7.299), we have
\[
\frac{1}{2\sqrt{\omega^2 + k^2}} I_+^\wedge(\omega) = \frac{e^{-i\omega a}}{2\sqrt{\omega^2 + k^2}} - g_+^\wedge(\omega), \quad 0 < \text{Im} \ \omega < k. \tag{7.301}
\]

This form is somewhat different from the Wiener-Hopf equation (7.286) with \( \mu = 0 \), because the first term on the right side is analytic in a strip instead of an upper half-plane. Far from being a drawback, we shall see that this is actually a help in our case—the simplification arising because of the common presence of the term \( \sqrt{\omega^2 + k^2} \) on both sides of the equation.

Before proceeding to solve (7.301) we should point out that it can be derived directly from the partial differential equation satisfied by \( u(x, y) \) without going through the intermediate step of the integral equation. This alternative procedure, whose details are left to the reader, is often simpler and more convenient. Turning to the solution of (7.301), we first write
\[
\frac{1}{\sqrt{\omega^2 + k^2}} = \frac{1}{\sqrt{\omega + ik}} \frac{1}{\sqrt{\omega - ik}}, \tag{7.302}
\]
where the branches are chosen as in Figure 7.20. Thus, in terms of \( \theta \) and \( \phi \), we have
\[
\sqrt{\omega + ik} = |\omega + ik|^{1/2} e^{i\theta/2}, \quad -\frac{\pi}{2} \leq \theta < \frac{3\pi}{2}, \tag{7.303}
\]
\[
\sqrt{\omega - ik} = |\omega - ik|^{1/2} e^{i\phi/2}, \quad -\frac{3\pi}{2} < \phi \leq \frac{\pi}{2}.
\]

We observe that with these definitions the product of \( \sqrt{\omega + ik} \) and \( \sqrt{\omega - ik} \) is real and positive for all real \( \omega \). Also \( \sqrt{\omega + ik} \) is analytic for \( \text{Im} \ \omega > -k \) and \( \sqrt{\omega - ik} \) for \( \text{Im} \ \omega < k \).

Inserting (7.302) in (7.301), we have
\[
\frac{I_+^\wedge(\omega)}{\sqrt{\omega + ik}} - \frac{e^{-i\omega a}}{\sqrt{\omega + ik}} = -2\sqrt{\omega - ik} g_+^\wedge(\omega). \tag{7.304}
\]

In this equation, the left side is analytic for \( \text{Im} \ \omega > -k \) and the right side for \( \text{Im} \ \omega < k \). Hence both sides are equal to an entire function and it might appear that our problem has been solved, but, unfortunately the presence of the term \( e^{-i\omega a} \) (which behaves badly in an upper half-plane) prevents us from determining the entire function. It is therefore necessary to write
\[
\frac{e^{-i\omega a}}{\sqrt{\omega + ik}} = a_+(\omega) + a_-(\omega),
\]
as in (7.291) so that we will be able to predict that $a_+(\omega)$ vanishes as $\text{Im} \, \omega \to +\infty$. We have

$$a_+(\omega) = \frac{1}{2\pi i} \int_{C_1} \frac{e^{-i\alpha a}}{(\alpha - \omega)\sqrt{\alpha + ik}} \, d\alpha,$$  \hspace{1cm} (7.305)

where $C_1$ is shown in Figure 7.21.

As a function of $\alpha$, the integrand in (7.305) is analytic in a lower half-plane except on the branch cut from $-ik$ to $-i\infty$. Moreover, the integrand is exponentially decaying in the lower half-plane, so that the contour $C_1$ is equivalent to $D$ by Cauchy's theorem and

$$a_+(\omega) = \frac{1}{2\pi i} \int_{D} \frac{e^{-i\alpha a}}{(\alpha - \omega)\sqrt{\alpha + ik}} \, d\alpha.$$
Setting $\alpha = iR$ on either side of the branch cut and using (7.303) with $\theta = 3\pi/2$ and $\theta = -\pi/2$ on the left and right sides of the cut, respectively, we obtain

$$a_+ (\omega) = \frac{e^{i\pi/4}}{\pi i} \int_k^\infty dR \frac{e^{-aR}}{(R - i\omega)(R - k)^{1/2}}, \quad \text{Im } \omega > -k.$$  

We evaluate $a_+ (\omega)$ for $\omega = ip$, $p$ real, $p > -k$, and then use analytic continuation for other values of $\omega$ satisfying $\text{Im } \omega > -k$. By Exercise 7.54 we then find

$$a_+ (ip) = -i \frac{e^{i\pi/4}}{(p + k)^{1/2}} e^{a p} \text{erfc} \left[a(p + k)\right]^{1/2},$$

where erfc is the complementary error function $1 - \text{erf}$. Now from the definition (7.303), we have, for $\theta = \pi/2$.

$$\sqrt{\omega + ik} = e^{i\pi/4}(p + k)^{1/2},$$

and therefore,

$$a_+ (\omega) = \frac{e^{-i\omega}}{\sqrt{\omega + ik}} \text{erfc} \left[\sqrt{a e^{-i\pi/4} \sqrt{\omega + ik}}\right].$$

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an expression which is analytic except on the branch cut in Figure 7.21, and hence is surely analytic for \( \text{Im} \, \omega > -k \). Equation (7.304) becomes

\[
\frac{I_{\tau}(\omega)}{\sqrt{\omega + ik}} - a_+(\omega) = a_-(\omega) - 2\sqrt{\omega - ik} \, g_\tau(\omega),
\]

the left side analytic for \( \text{Im} \, \omega > -k \), the right side for \( \text{Im} \, \omega < k \). Both sides are equal to the same entire function \( E(\omega) \). Since \( I_{\tau} \) and \( a_+ \) tend to 0 as \( \text{Im} \, \omega \to +\infty \), \( E(\omega) \to 0 \) in the upper half-plane. Since \( a_- \) and \( g_\tau(\omega) \) tend to 0 as \( \text{Im} \, \omega \to -\infty \), \( |E(\omega)| = 0(|\omega|^{1/2}) \) as \( \text{Im} \, \omega \to \infty \). These two conditions on \( E \) imply that \( E \equiv 0 \). Therefore,

\[
I_{\tau}(\omega) = \sqrt{\omega + ik} \, a_+(\omega) = e^{-ia_0} \, \text{erfc} \left[ \sqrt{a} \, e^{-in/4} \sqrt{\omega + ik} \right], \tag{7.306}
\]

which is analytic, \( \text{Im} \, \omega > -k \). We now find \( I(x) \) by inverting (7.306); thus

\[
I(x) = \frac{1}{2\pi} \int_{c_1} e^{-ia(x+a)} \, \text{erfc} \left[ \sqrt{a} \, e^{-in/4} \sqrt{\omega + ik} \right] d\omega,
\]

and again we deform \( C_1 \) into \( D \) to obtain

\[
I(x) = -\frac{i}{\pi^{3/2}} \, e^{-k(x+a)} \int_0^\infty e^{-R(x+a)} dR \int_{-i\sqrt{aR}}^{i\sqrt{aR}} e^{-z^2} \, dz.
\]

Setting \( z = iv \) and changing the order of integration (paying due attention to the needed modification in the limits), we find

\[
I(x) = \frac{2}{\pi^{3/2}} \, e^{-k(x+a)} \int_0^\infty e^{-u^2/a} \, du = \frac{\sqrt{a} \, e^{-k(x+a)}}{\pi \, \sqrt{x(x+a)}}. \tag{7.307}
\]

If we let \( k^2 = -\lambda \) in (7.295), with \( \sqrt{\lambda} \) having positive imaginary part, then \( k = -i\sqrt{\lambda} \), and

\[
I(x) = \frac{\sqrt{a}}{\pi} \left[ \frac{e^{i\sqrt{\lambda} \, (x+a)}}{\sqrt{x(x+a)}} \right], \tag{7.308}
\]

which is valid, by analytic continuation in \( \lambda \), for all \( \lambda \) not in \([0, \infty)\). For \( \lambda \) real, positive, the formula (7.308) is still valid by the principle of limiting absorption, as long as we understand \( \sqrt{\lambda} \) to be the positive square root.

**Exercises**

7.44 Equation (7.250) is an integral equation of the first kind for the current on an obstacle. One can also obtain an equation of the second kind by returning to (7.249) and taking the derivative in the \( v \) direction (here \( v \) is the outward normal at some point \( s \) on \( \sigma \)). Then

\[
\frac{\partial u_s}{\partial \nu}(x) = -\int_{\sigma} \frac{\partial}{\partial \nu} \left[ E(x | \xi) \right] I(\xi) dS_{\xi},
\]
may be interpreted as the normal derivative of the potential of a simple layer on $\sigma$. As in (6.43) we can show that
\[
\lim_{x \to s^+} \frac{\partial u_x}{\partial y} (x) = + \frac{1}{2} I(s) - \int_{\sigma} \frac{\partial}{\partial y} [E(s \mid \xi)] I(\xi) dS_\xi,
\]
and hence
\[
\frac{1}{2} I(s) - \frac{\partial u_i}{\partial y} (s) = - \int_{\sigma} \frac{\partial}{\partial y} [E(s \mid \xi)] I(\xi) dS_\xi.
\]
which is an equation of the second kind for the current $I$.

7.45 If $\sigma$ is an open shell, as in Figure 6.5, show that (7.309) reduces to
\[
I_+(s) - I_-(s) - 2 \frac{\partial u_i}{\partial y} (s) = -2 \int_{\sigma} \frac{\partial}{\partial y} [E(s \mid \xi)] I(\xi) dS_\xi,
\]
where $I = I_+ + I_-$. Although this is not an integral equation for $I$, show that in conjunction with (7.250), it enables us to solve for the individual currents $I_+$ and $I_-$ on either side of the shell.

7.46 Consider scattering of a plane wave by a bounded three-dimensional obstacle with the boundary condition of vanishing normal derivative. Derive the reciprocity principle and express the scattering cross section in terms of the forward scattered amplitude. Discuss the possible integral equations on the boundary which would permit us to solve the scattering problem.

7.47 Consider scattering of a plane wave by an infinite cylinder with the boundary condition $u = 0$. Derive the two-dimensional analogs of (7.253), (7.254), and (7.260).

7.48 Using separation of variables in polar coordinates, find a series for the scattering cross section of an infinite circular cylinder.

7.49 A unit point source is located at $(-a, 0, 0)$ in front of a half-plane whose equation is $x > 0, y = 0, -\infty < z < \infty$. The field $v(x, y, z)$ satisfies the Helmholtz equation
\[
-\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial y^2} - \frac{\partial^2 v}{\partial z^2} + k^2 v = \delta(x + a)\delta(y)\delta(z) + v(x, 0+, z) = v(x, 0-, z) = 0,
\]
$0 < x < \infty, -\infty < z < \infty$.

Take a Fourier transform on $z$ to reduce this problem to (7.295) with $k^2$ replaced by $(k^2 + \alpha^2)^{1/2}$, where $\alpha$ is the transform variable. Use (7.307) to show that the current
\[
I = \frac{\partial v}{\partial y} (x, 0+, z) - \frac{\partial v}{\partial y} (x, 0-, z)
\]
is given by
\[ a^{1/2}k \frac{1}{\pi^2 x^{1/2} [z^2 + (x + a)^2]^{1/2}} K'_0 \{ k [(x + a)^2 + z^2]^{1/2} \}. \]

For the electrostatic case (\( k \to 0 \)) of a point charge in front of a grounded half-plane, show that the total charge density (which corresponds to \(-I\)) is
\[ a^{1/2}k \frac{1}{\pi^2 x^{1/2} [z^2 + (x + a)^2]}. \]

Verify that the total charge on the half-plane is \(-1\).

7.50 A plane wave \( e^{i\sqrt{\lambda}z} \) is incident from \( z < 0 \) on an infinite screen \( \sigma \) located in the plane \( z = 0 \). The screen contains an aperture \( A \). The screen and aperture together cover the entire plane \( z = 0 \). The boundary condition on the screen is that the total field \( u \) vanishes. The current on the screen is defined as
\[ I(x, y) = \frac{\partial u}{\partial z} (x, y, 0^+) - \frac{\partial u}{\partial z} (x, y, 0^-) \]
and satisfies the integral equation (7.250) with \( u_i(s) = 1 \).

This integral equation is not entirely satisfactory, since \( I \) does not tend to 0 at remote points on the screen. On physical grounds it is preferable to split the field as
\[ u = \begin{cases} e^{i\sqrt{\lambda}z} - e^{-i\sqrt{\lambda}z} + v, & z < 0, \\ v, & z > 0. \end{cases} \]

In the case of a solid screen with no aperture, \( v \) would be zero everywhere, so that for a bounded aperture \( v \) may be regarded as a perturbation on the field that would exist if no aperture were present. Define
\[ J(x, y) = \frac{\partial v}{\partial z} (x, y, 0^+) - \frac{\partial v}{\partial z} (x, y, 0^-), \]
and express \( v \) in terms of \( J \); then obtain the integral equation for \( J \):
\[ \int_{\sigma} E(x, y, 0 | \xi, \eta, 0) J(\xi, \eta) d\xi \, d\eta = 2i\sqrt{\lambda} \int_{A} E(x, y, 0 | \xi, \eta, 0) d\xi \, d\eta, \quad x, y \text{ on } \sigma. \]

By using a Green's function with odd symmetry about \( z = 0 \), show that the field everywhere can be expressed in terms of the field in the aperture.

7.51 Problem (7.50) with the boundary condition \( \partial u/\partial n = 0 \) on the screen. Derive an expression for the field everywhere in terms of the normal
derivative of the field in the aperture. Characterize this normal derivative by an integral equation. Compare this result with diffraction by a thin disk of the shape of the aperture (with vanishing field on the disk.)

7.52 Consider steady diffusion in an absorbing medium. The absorption coefficient is the constant \( k^2 \) outside the bounded region \( R \) and a function \( k^2 - k_0^2(x) \) inside \( R \). A point source is located at \( x_0 \) in the exterior of \( R \). The concentration \( u \) satisfies

\[
- \nabla^2 u + k^2 u = \delta(x - x_0), \quad x \text{ outside } R \\
- \nabla^2 u + [k^2 - k_0^2(x)]u = 0, \quad x \text{ inside } R.
\]

Express \( u \) everywhere in terms of its values inside \( R \) and derive an integral equation for \( u(x), x \text{ in } R \).

\[\text{FIGURE 7.22}\]

7.53 Figure 7.22 is a cross-sectional view of two parallel plates with a screen obstructing in part the plane \( z = 0 \). The boundary condition on the plates and on the screen is that the normal derivative of the field vanishes. If the screen were absent, the plane wave \( e^{i\sqrt{\lambda}z} \) could propagate in the channel. In the presence of the screen this plane wave is diffracted. The total field \( u \) can be written

\[u = e^{i\sqrt{\lambda}z} + u_s,\]

where \( u_s(x, z) \) satisfies

\[- \nabla^2 u_s - \lambda u_s = 0; \quad \frac{\partial u_s}{\partial x}(0, z) = \frac{\partial u_s}{\partial x}(a, z) = 0; \]

\[\frac{\partial u_s}{\partial z}(x, 0) = -i\sqrt{\lambda}, \quad \frac{a}{2} < x < a,\]
and \( u_s \) decays exponentially for \( |z| = \infty \). We have assumed that \( \sqrt{\lambda} \) has positive imaginary part. By expanding \( u_s \) in a cosine series for \( z > 0 \) and \( z < 0 \), obtain an integral equation for the unknown value of \( \partial u / \partial z \) in \( 0 < x < a/2, \ z = 0 \). Derive the same integral equation by a Green's function technique.

7.54 Consider

\[
F(a) = \int_k^\infty dR \frac{e^{-a(R+p)}}{(R+p)(R-k)^{1/2}}.
\]

By calculating \( dF/da \) and integrating, show that

\[
F(a) = \frac{\pi}{\sqrt{p+k}} \left[ 1 - \text{erf}\sqrt{a(p+k)} \right].
\]

7.55 Show that the integral equation (7.292) can be reduced to

\[
u'' - 9u = 4, \quad 0 < x < \infty, \quad u(0) - u'(0) = 4.
\]

The only solution for which the integral in (7.292) converges is

\[
u(x) = \frac{4}{9} \left[ 1 + 2e^{-3x} \right],
\]

in agreement with the result obtained by the Wiener-Hopf method.

7.56 When we consider the homogeneous system (7.295) with a plane wave \( e^{i\phi} \) incident from the positive \( y \) direction, the integral equation (7.297) becomes

\[
1 = \int_0^\infty \frac{1}{2\pi} K_0(k|x - \xi|)I(\xi)d\xi, \quad x > 0.
\]

Defining

\[
g^-(x) = \begin{cases} 
\int_0^\infty \frac{1}{2\pi} K_0(k|x - \xi|)I(\xi)d\xi, & x < 0, \\
0, & x > 0
\end{cases}
\]

the transformed integral equation becomes

\[
\frac{1}{2\sqrt{\omega^2 + k^2}} I^+ = \frac{i}{\omega} + g^-, \quad 0 < \text{Im} \omega < k.
\]

Show by the Wiener-Hopf method that

\[
I^+ = 2ie^{-i\pi/4}k^{1/2} \frac{\sqrt{\omega + ik}}{\omega},
\]

\[
I(x) = \frac{\sqrt{k}}{\pi} \int_k^\infty \frac{e^{-xR}}{R} \sqrt{R-k} dR.
\]
Express this last integral in terms of the error function.

7.57 Consider the mixed boundary value problem

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - k^2 u = 0, \quad -\infty < x < \infty, \quad y > 0
\]

\[
\frac{\partial u}{\partial y} (x, 0) = 0, \quad -\infty < x < 0; \quad u(x, 0) = e^{-\varepsilon x}, \quad 0 < x < \infty.
\]

Use a Green's function whose normal derivative vanishes on \( y = 0 \) to express the solution in terms of the unknown values of \( \partial u/\partial y(x, 0) \), \( 0 < x < \infty \). Solve for this unknown by the Wiener-Hopf method. Consider also the limiting cases \( k = 0 \) and \( \varepsilon = 0 \).
Chapter 8

VARIATIONAL
AND RELATED METHODS

8.1 INTRODUCTION

The solution \( u(x) \) of a boundary value problem can often also be characterized as the function which yields a maximum value (or sometimes a minimum or stationary value) to a related functional \( F(u) \). The maximum of \( F \) is to be sought not among all functions but only within a fairly general class of functions known as \textit{admissible functions}. The class of admissible functions may of course change from one problem to another but usually will include all functions satisfying some or all boundary conditions of the boundary value problem and certain continuity conditions. One of the important mathematical applications of the variational principle is in proving the existence of a solution to the boundary value problem, but we shall not be concerned to any great extent with this aspect of the theory. Instead we assume (or prove by an independent method) that the boundary value problem has a solution and then use the maximum principle to construct an approximation \( \tilde{u} \) to \( u \). The principal method for constructing \( \tilde{u} \) is the \textit{Ritz-Rayleigh procedure}, which consists of finding a constrained maximum of \( F \), that is, finding the maximum of \( F \) not among all admissible functions but only within a particularly simple subset of admissible functions. There is another feature of the maximum principle, which is of great importance in applications. The numerical quantity \( F(u) \) is in itself of considerable interest, because it often represents a kind of a weighted average of \( u \); in fact, in many examples \( F \) is proportional to the energy of the physical system under investigation and we may be satisfied with finding a reliable approximation to the number \( F(u) \). The quantity \( F(\tilde{u}) \) obtained from the Ritz-Rayleigh procedure will usually provide a satisfactory estimate of \( F(u) \) even if the approximation \( \tilde{u} \) to \( u \) is not
entirely acceptable. The phenomenon is similar to what happens for a differentiable function of a real variable: If \( F(x) \) has a maximum at \( x = x_0 \), then the graph of \( F(x) \) is flat at \( x = x_0 \) and the value of \( F \) at points \( x \) near \( x_0 \) is very close to the value of \( F \) at \( x_0 \). If \( F \) has a second derivative at \( x_0 \), then \( |F(x) - F(x_0)| \) is of order \( \varepsilon^2 \) if \( |x - x_0| \) is of order \( \varepsilon \).

A simple example will serve to illustrate some of these ideas. Consider the static deflection \( u(x) \) of a taut string under unit tension, subject to a given continuous transverse pressure \( f(x) \). If the string is fixed at its ends, \( x = 0 \) and \( x = l \), the deflection \( u(x) \) satisfies the system

\[
- \frac{d^2 u}{dx^2} = f(x), \quad 0 < x < l; \quad u(0) = u(l) = 0. \tag{8.1}
\]

It is, of course, well known that this boundary value problem has one and only one solution \( u(x) \). The strain energy in the string is easily calculated and we find

\[
W_s = \frac{1}{2} \int_0^l \left( \frac{du}{dx} \right)^2 \, dx.
\]

The work done by the pressure \( f(x) \) to bring the string from its undeflected state to its deflected one is

\[
\int_0^l f(x)u(x)\,dx.
\]

Thus the total potential energy \( W \) of the string in its deflected state is

\[
W(u) = \frac{1}{2} \int_0^l \left( \frac{du}{dx} \right)^2 \, dx - \int_0^l f(x)u(x)\,dx. \tag{8.2}
\]

Substitution of the solution of (8.1) in (8.2) yields the exact potential energy of the deflected string. Moreover, from the theorem of minimum potential energy in mechanics, we know that if we substitute in (8.2) any other function \( v(x) \), having a continuous second derivative, and satisfying \( v(0) = v(l) = 0 \), we get a larger value for \( W \). The requirement that \( v \) be twice differentiable can be relaxed and we state the theorem as follows: Of all functions \( v(x) \) having a piecewise continuous derivative and satisfying \( v(0) = v(l) = 0 \), the one which minimizes the functional

\[
W(v) = \frac{1}{2} \int_0^l \left( \frac{dv}{dx} \right)^2 \, dx - \int_0^l f(x)v(x)\,dx \tag{8.3}
\]

is the function \( u \) which is the solution of (8.1).

The proof is simple. If \( v \) is admissible in (8.3), we can write \( v = u + h \), where \( h \) has a piecewise continuous derivative and obeys the conditions \( h(0) = h(l) = 0 \). Then

\[
W(u + h) = W(u) + \int_0^l u'h' \, dx + \frac{1}{2} \int_0^l (h')^2 \, dx - \int_0^l f h \, dx,
\]
and integrating the second term by parts and using (8.1),

\[ W(u + h) = W(v) = W(u) + \int_0^l (h')^2 \, dx. \]

The last term is positive unless \( h \) is a constant. Since \( h(0) = 0 \), the constant
must be 0. Therefore if \( h \) is not identically 0,

\[ W(v) > W(u). \]

Now let us show how the Ritz-Rayleigh procedure can be used to find an
approximate solution \( \bar{u} \) of (8.1) and correspondingly an approximation \( W(\bar{u}) \)
to the potential energy. We know that \( u \) [the correct solution of (8.1)] is the
function which yields the minimum of (8.3) among all admissible functions \( v(x) \),
where an admissible function has a piecewise continuous derivative and vanishes at \( x = 0 \) and \( x = l \). The Ritz-Rayleigh procedure minimizes (8.3) within a smaller class of functions \( E_n \), where \( E_n \) is the \( n \)-dimensional
space of linear combinations of the \( n \) independent admissible functions \( v_1(x), \ldots, v_n(x) \).
Every function \( v \) in \( E_n \) can be written

\[ \sum_{i=1}^n c_i v_i(x), \quad (8.4) \]

where \( c_1, \ldots, c_n \) are real constants. Then for any choice of \( \{c_i\} \), we have

\[ W\left(\sum_{i=1}^n c_i v_i\right) \geq W(u). \]

We now choose \( c_1, \ldots, c_n \) so that the left side is minimized. This leads to

\[ \frac{\partial W\left(\sum_{i=1}^n c_i v_i\right)}{\partial c_k} = 0, \quad k = 1, \ldots, n. \]

From (8.3) we obtain explicitly

\[ \sum_{i=1}^n c_i \int_0^l v_i' v_k' \, dx = \int_0^l f(x) v_k(x) \, dx, \quad k = 1, \ldots, n. \quad (8.5) \]

If we solve these \( n \) linear algebraic equations for \( c_1, \ldots, c_n \) and substitute in
(8.4) we obtain the Ritz-Rayleigh approximation \( \bar{u} \). The corresponding
approximate value of the potential energy is

\[ W(\bar{u}) = W\left(\sum_{i=1}^n c_i v_i\right). \quad (8.6) \]

As a numerical example consider (8.1) for the case \( f(x) = \sin(\pi x/l) \). Let us
temporarily forget that we know how to solve (8.1) explicitly in this particular
case. Instead we shall use the Ritz-Rayleigh procedure with a single term $c_1v_1(x)$, where $v_1(x) = x(l - x)$. Then, from (8.5), we find

$$c_1 = \frac{\int_0^l x(l - x) \sin \frac{\pi x}{l} \, dx}{\int_0^l (l - 2x)^2 \, dx} = \frac{12}{\pi^3}.$$ 

Thus

$$\bar{u} = \frac{12}{\pi^3} x(l - x),$$

$$W(\bar{u}) = -\frac{24l^3}{\pi^6}.$$ 

For purposes of comparison, we now conveniently remember the exact solution to (8.1), which is found by elementary methods to be

$$u = \frac{l^2}{\pi^2} \sin \frac{\pi x}{l}.$$ 

The exact potential energy is

$$W(u) = -\frac{l^3}{4\pi^2}.$$ 

At $x = l/2$, we have $\bar{u} = (3/\pi^3)l^2$, which differs from the exact value $l^2/\pi^2$ by 4.5 per cent. On the other hand, $W(\bar{u})$ differs from $W(u)$ by only 1 per cent.

Starting with Section 8.3, we shall consider these extremal principles on a systematic basis, but first we review and extend some concepts from Hilbert space (see Volume I, Chapter 2).

8.2 BEST APPROXIMATION IN A SUBSPACE

Let $f$ be an arbitrary element of the complex Hilbert space $\mathcal{F}$ and let $M$ be an $n$-dimensional closed linear manifold in $\mathcal{F}$. We look for the element $g$ in $M$ which is closest to $f$, that is, for which $\|g - f\|$ is minimum. This element $g$ is unambiguously defined and is the projection of $f$ on $M$; we denote the element $g$ by $Pf$, where $P$ is the projection operator on $M$. An explicit expression for $Pf$ is easily obtained by introducing an orthonormal basis $\varphi_1, \ldots, \varphi_n$ in $M$. Then, as we have seen in Section 2.4, we can write

$$Pf = \sum_{k=1}^n \langle f, \varphi_k \rangle \varphi_k.$$ 

(8.7)

Frequently we will want to express $Pf$ in terms of a basis in $M$ which is not orthonormal. Let $v_1, \ldots, v_n$ be an arbitrary basis in $M$; since $Pf$ is an element
in \( M \), we can write

\[
Pf = \sum_{k=1}^{n} a_k v_k.
\]

To calculate the coefficients we observe that the element \( f - Pf \) must be orthogonal to \( M \), hence to each of the elements \( v_1, \ldots, v_n \). The coefficients \( a_1, \ldots, a_n \) are therefore unambiguously determined from the \( n \) linear algebraic equations

\[
\langle f, v_j \rangle = \sum_{k=1}^{n} a_k \langle v_k, v_j \rangle, \quad j = 1, \ldots, n.
\]

(8.9)

To solve these equations for \( \{a_k\} \) we introduce the matrix \( (b_{ij}) \), with \( b_{ij} = \langle v_i, v_j \rangle \). Clearly \( b_{ij} = b_{ji} \), so that the matrix is symmetric; moreover, since the \( \{v_i\} \) are independent, the matrix is nonsingular and its inverse is also symmetric. We therefore find

\[
a_k = \sum_{j=1}^{n} c_{jk} \langle f, v_j \rangle,
\]

where \( (c_{ij}) \) is the inverse matrix of \( (b_{ij}) \). Substitution in (8.8) yields

\[
Pf = \sum_{i,j=1}^{n} c_{ij} \langle f, v_i \rangle v_j.
\]

(8.10)

We note some important features of the operator \( P \). First, \( P \) is symmetric; that is, if \( f \) and \( g \) are two elements of \( \mathcal{A} \),

\[
\langle Pf, g \rangle = \langle f, Pg \rangle.
\]

The proof is simple; using an orthonormal basis \( \varphi_1, \ldots, \varphi_n \) in \( M \) and (8.7), we have

\[
\langle Pf, g \rangle = \sum_{i=1}^{n} \langle f, \varphi_i \rangle \langle \varphi_i, g \rangle
\]

\[
\langle f, Pg \rangle = \sum_{i=1}^{n} \langle f, \varphi_i \rangle \langle g, \varphi_i \rangle = \langle Pf, g \rangle.
\]

Furthermore, \( P \) is a nonnegative operator; that is, \( \langle Pf, f \rangle \geq 0 \) for each \( f \) in \( \mathcal{A} \). Indeed,

\[
\langle Pf, f \rangle = \sum_{i=1}^{n} \langle f, \varphi_i \rangle \langle \varphi_i, f \rangle = \sum_{i=1}^{n} |\langle f, \varphi_i \rangle|^2.
\]

We leave it to the reader to show that \( P^2 = P \). Of course, if we wish we may regard \( P \) as an operator defined on \( M \) rather than on \( \mathcal{A} \); then \( P \) is just the identity operator:

\[
Pf = f, \quad \text{if} \ f \in E_n.
\]
8.3 MAXIMUM THEOREM

In what follows, we shall be concerned mostly with symmetric, positive operators defined on a linear manifold $D_A$ dense in $\mathcal{A}$, where $\mathcal{A}$ is a complex Hilbert space. Since $A$ is symmetric on $D_A$, we have

$$\langle Au, v \rangle = \langle u, Av \rangle, \quad \text{for all } u, v \text{ in } D_A.$$ 

For a symmetric operator, $\langle Au, u \rangle$ is always real. By the definition of a positive operator,

$$\langle Au, u \rangle > 0, \quad \text{for all } u \neq 0 \text{ in } D_A.$$

We now investigate the inhomogeneous equation

$$Au = f, \quad u \in D_A,$$

(8.11)

where $f$ is a given element in $\mathcal{A}$, and $A$ is a symmetric, positive operator.

**Theorem 1.** Equation (8.11) has at most one solution.

**Proof.** Suppose $u_1$ and $u_2$ are two solutions of (8.11) and let $z = u_1 - u_2$. Then $z \in D_A$ and $Az = 0$, so that $\langle Az, z \rangle = 0$. This in turn implies that $z = 0$, since $A$ is positive.

We are interested in a number of questions relating to (8.11). First, does the equation have a solution? If so, how can we calculate it, at least approximately? Even when it is too difficult to find an adequate approximation to the solution $u$, we may be satisfied with estimating accurately a certain numerical quantity of physical interest associated with (8.11). To be specific, the number

$$\alpha = \langle f, u \rangle$$

is often of special importance, as we shall see later in examples.

We observe from (8.11) that

$$\alpha = \langle f, u \rangle = \langle Au, u \rangle = \langle u, Au \rangle = \langle u, f \rangle$$

(8.12)

so that $\alpha$ is a positive number. We also have

$$\alpha = \langle f, u \rangle + \langle u, f \rangle - \langle Au, u \rangle.$$ 

(8.13)

Partial answers to the questions we have raised are provided by the following maximum theorem.

**Theorem 2 (Maximum Theorem).** Let

$$F(v) = \langle f, v \rangle + \langle v, f \rangle - \langle Av, v \rangle.$$ 

(8.14)

Then if (8.11) has a solution $u$, $u$ is the one and only element in $D_A$ which maximizes $F$. Conversely, if among the elements in $D_A$ there exists one which maximizes $F$, then it is the solution of (8.11).
Proof. (a) Let \( u \) be the solution of (8.11). We must show that if \( v \in D_A \) and \( v \neq u \), then \( F(v) < F(u) \). Setting \( u - v = h \), we have

\[
F(u) - F(v) = \langle f, h \rangle + \langle h, f \rangle - \langle Au, h \rangle + \langle Ah, h \rangle \\
= \langle f - Au, h \rangle + \langle h, f - Au \rangle + \langle Ah, h \rangle \\
= \langle Ah, h \rangle.
\]

Since \( h \neq 0 \), \( \langle Ah, h \rangle > 0 \) and \( F(u) > F(v) \).

(b) Let (8.14) have a maximum for an element \( u \) in \( D_A \). We must show that \( u \) is a solution of (8.11). By assumption, for any real number \( \varepsilon \) and any element \( \eta \) in \( D_A \), we have

\[
F(u) \geq F(u + \varepsilon \eta).
\]

With \( \eta \) fixed, the right side is a function of the real parameter \( \varepsilon \) and has a maximum at \( \varepsilon = 0 \). Therefore,

\[
\left[ \frac{d}{d\varepsilon} F(u + \varepsilon \eta) \right]_{\varepsilon = 0} = 0.
\]

Carrying out the indicated operation, we find

\[
\langle f, \eta \rangle + \langle \eta, f \rangle - \langle A\eta, u \rangle - \langle Au, \eta \rangle = 0,
\]

or

\[
\langle f - Au, \eta \rangle + \langle \eta, f - Au \rangle = 0.
\]

This last equation must hold for any \( \eta \) in \( D_A \). Since \( D_A \) is a linear manifold, \( i\eta \) also belongs to \( D_A \). Substituting \( i\eta \) for \( \eta \), we obtain

\[
-\langle f - Au, \eta \rangle + \langle \eta, f - Au \rangle = 0.
\]

Subtracting this equation from the previous one, we have

\[
\langle f - Au, \eta \rangle = 0, \quad \text{for all } \eta \text{ in } D_A.
\]

Since \( D_A \) is dense in \( \mathcal{A} \), it follows that

\[
f - Au = 0,
\]

which completes the proof.

Remarks. 1. In most applications, we shall only use the first part of the maximum theorem. Thus we shall assume (or prove by other means) that (8.11) has a solution \( u \). Since by (8.13), we have \( \alpha = F(u) \), our theorem then states

\[
\alpha = F(u) = \max_{v \in D_A} F(v); \quad \text{maximum occurs for } v = u. \quad (8.15)
\]

2. We can restate Theorem 2 as a minimum principle by using the functional \(-F(v)\) instead of \(F(v)\). In this form the principle coincides, apart from a factor of 2, with the principle of minimum potential energy. In practice nothing is gained by this reformulation; in one case we have
a minimum theorem for $-\alpha$, in the other a maximum theorem for $\alpha$, and obviously any upper bound for $-\alpha$ automatically yields, with a change of sign, a lower bound for $\alpha$. If one wants a minimum theorem for $\alpha$, one must use the methods of Section 8.5.

3. If $A$ is a positive symmetric operator on a finite-dimensional space $E_n$, Theorem 2 is still true (with $D_A = E_n$) and the proof is identical with the one above. Moreover, in this case we know that $Au = f$ always has a solution, since the operator $A$ is nonsingular. Therefore there is one and only one element which maximizes (8.14) and that element is the solution of $Au = f$.

4. If $A$ is a negative symmetric operator, Theorem 2 holds with the word maximum replaced everywhere by minimum.

5. If $A$ is a real operator and $f$ is real, then we can restrict ourselves to real functions in Theorem 2 and use the simpler form

$$F(v) = 2\langle f, v \rangle - \langle Av, v \rangle.$$  \hfill (8.16)

As an illustration of the maximum theorem, let us return to the boundary value problem (8.1). Let $A$ be the operator $-d^2/dx^2$ defined on the domain $D_A$ of real functions having a continuous second derivative and satisfying the boundary conditions $u(0) = u(l) = 0$. We show that $A$ is symmetric on $D_A$. Indeed,

$$\langle Av, w \rangle = - \int_0^l w'' v \, dx = - \int_0^l w'' v \, dx + [w' v - w v']_0,$$

and, since $v, w$ satisfy the boundary conditions,

$$\langle Av, w \rangle = \langle v, Aw \rangle.$$

To show that $A$ is positive on $D_A$, we observe that

$$\langle Av, v \rangle = - \int_0^l v'' v \, dx = \int_0^l (v')^2 \, dx - vv',$$

and since $v$ satisfies the boundary conditions,

$$\langle Av, v \rangle = \int_0^l (v')^2 \, dx \geq 0.$$

We can have $\langle Av, v \rangle = 0$ only if $v' = 0$, that is, $v = \text{constant}$. By the boundary conditions on $v$, this implies $v = 0$. Therefore, if $v$ is not the 0 function, $\langle Av, v \rangle > 0$. We can therefore apply our maximum theorem, and find

$$\max_{v \in D_A} \left[ 2 \int_0^l fu \, dx - \int_0^l (u')^2 \, dx \right] = 2 \int_0^l fu \, dx - \int_0^l (u')^2 \, dx.$$

It therefore follows that

$$\min_{v \in D_A} \left[ \frac{1}{2} \int_0^l (v')^2 \, dx - \int_0^l fv \, dx \right] = \frac{1}{2} \int_0^l (u')^2 \, dx - \int_0^l fu \, dx,$$

which is just (8.3).
There is an alternative form for the maximum theorem which deserves special mention. The advantage of the new formulation is that it is scale-independent; that is, it involves a functional $R$ with the property $R(cv) = R(v)$, for any constant $c \neq 0$.

**Theorem 3. (Schwinger-Levine Principle).** Let

$$R(v) = \frac{|\langle f, v \rangle|^2}{\langle Av, v \rangle}, \quad v \neq 0. \quad (8.17)$$

Then $R(v)$ and $F(v)$ have the same maximum value.

**Proof.** Since $R(u) = \alpha$, it is clear that max $R \geq$ max $F$. Let $z$ be a function which yields the maximum value for $R$; then so does $cz$, where $c$ is any nonzero constant. If we choose $c = \langle f, z \rangle / \langle Az, z \rangle$, then

$$F(cz) = \bar{c}\langle f, z \rangle + c\langle z, f \rangle - c\bar{c}\langle Az, z \rangle$$

$$= \frac{|\langle f, z \rangle|^2}{\langle Az, z \rangle} = R(z).$$

Therefore, max $F \geq$ max $R$. In conjunction with the previous inequality, we have max $F = \max R$, which proves the theorem. The equivalent form of (8.15) is

$$\alpha = R(u) = \max_{v \in D_A} R(v); \quad \text{maximum occurs for } v = cu. \quad (8.18)$$

### 8.4 RITZ-RAYLEIGH METHOD

We want to obtain an approximate solution to (8.11) by using the maximum theorem (Theorem 2); we shall assume that (8.11) has a solution, so that there is a single function $u$ which maximizes $F(v)$ and that function is the solution of (8.11). Our approximation method is based on taking the maximum of $F$ not on the set of all functions in $D_A$ but rather the restricted maximum of $F$ on an $n$-dimensional subspace $E_n$ of $D_A$. Then we have

$$\max_{v \in E_n} F(v) \leq \max_{v \in D_A} F(v) = F(u);$$

our goal is to evaluate the left side exactly and thus obtain a lower bound to $\alpha = F(u)$. At the same time, the function which yields the maximum on $E_n$ will be considered an approximation to $u$.

We define $P_n$ as the projection operator on $E_n$. For an arbitrary element $z$ in $D_A$, we can use either (8.7) or (8.10) to get an explicit representation of $P_nz$. In any event, if $v \in E_n$, then $P_nv = v$ and,

$$F(v) = \langle f, v \rangle + \langle v, f \rangle - \langle Av, v \rangle = \langle f, P_nv \rangle + \langle P_nv, f \rangle - \langle Av, P_nv \rangle.$$ 

Since $P_n$ is symmetric,

$$F(v) = \langle P_nf, v \rangle + \langle v, P_nf \rangle - \langle P_nAv, v \rangle, \quad v \in E_n. \quad (8.19)$$
The operator $P_nA$ may be regarded as an operator on $E_n$ and as such is called the part of $A$ in $E_n$. This operator is symmetric and positive; indeed, if $z$ and $w$ are in $E_n$,

$$
\langle P_nAz, w \rangle = \langle Az, P_nw \rangle = \langle Az, w \rangle = \langle z, Aw \rangle
$$

$$
= \langle P_nz, Aw \rangle = \langle z, P_nAw \rangle,
$$

and, if $z \neq 0$,

$$
\langle P_nAz, z \rangle = \langle Az, P_nz \rangle = \langle Az, z \rangle > 0.
$$

Let us consider the functional (8.19) for $v \in E_n$. This is just the maximizing functional associated with the $n$-dimensional equation

$$
P_nAu_n = P_nf, \quad u_n \in E_n. \tag{8.20}
$$

We may then apply Theorem 2 and Remark 3 following that theorem: Among all functions in $E_n$, the one and only solution $u_n$ of (8.20) maximizes $F(v)$ as given by (8.19); that is,

$$
F(u_n) = \max_{v \in E_n} (\langle P_nf, v \rangle + \langle v, Pf \rangle - \langle P_nAv, v \rangle). \tag{8.21}
$$

The Ritz-Rayleigh procedure, which consists of finding the function in $E_n$ which maximizes the right side of (8.21), is therefore exactly equivalent to solving (8.20). But (8.20) has a simple interpretation quite independent of any extremal principle. In fact, all we have to do is project both sides of (8.11) on $E_n$ and then look for a solution belonging to $E_n$. From this point of view, nothing prevents us from using (8.20) even if $A$ is not positive, possibly even when $A$ is nonlinear. In this context (8.20) is often called the Galerkin equation.

The vector equation (8.20) can be translated into a set of $n$ algebraic equations in $n$ unknowns as follows. Let $v_1, \ldots, v_n$ be the basis for $E_n$; then, since $u_n \in E_n$, we can write

$$
u_n = \sum_{k=1}^{n} a_kv_k, \tag{8.22}
$$

where the coefficients $a_k$ must be calculated from (8.20). Taking the inner product of both sides of (8.20) with respect to $v_j$, we find

$$
\langle Au_n, v_j \rangle = \langle f, v_j \rangle,
$$

or

$$
\sum_{k=1}^{n} a_k \langle Av_k, v_j \rangle = \langle f, v_j \rangle, \quad j = 1, \ldots, n. \tag{8.23}
$$

The $n$ linear equations (8.23) determine $a_1, \ldots, a_n$ unambiguously; we then substitute in (8.22) to find the one and only solution of (8.20). The equations (8.23) are referred to as the Ritz-Rayleigh or Galerkin equations. If our
principal interest is in the estimation of \( \alpha \), we should in theory substitute \( u_n \) as calculated from (8.22) and (8.23) into the expression (8.14) or (8.19) for \( F \). It turns out that a simplifying feature of the Ritz-Rayleigh approximation is that the three quantities \( \langle f, u_n \rangle, \langle u_n, f \rangle, \) and \( \langle Au_n, u_n \rangle \) are all equal to one another. To show this is easy from (8.20); we have

\[
\langle P_nf, u_n \rangle = \langle f, u_n \rangle = \langle Pu_n, u_n \rangle = \langle Au_n, u_n \rangle = \langle u_n, Pf \rangle.
\]

Any approximate solution \( \tilde{u} \) of (8.11) for which

\[
\langle f, \tilde{u} \rangle = \langle \tilde{u}, f \rangle = \langle A\tilde{u}, \tilde{u} \rangle
\]

is said to satisfy the reciprocity principle. We observe from (8.12) that the exact solution of (8.11) satisfies the reciprocity principle; in addition, we have shown in (8.24) that the Ritz-Rayleigh approximation always satisfies the reciprocity principle. Thus

\[
F(u_n) = \langle f, u_n \rangle + \langle u_n, f \rangle - \langle Au_n, u_n \rangle = \langle f, u_n \rangle.
\]

Hence if our interest is in obtaining an estimate for \( \alpha \), we need only substitute \( u_n \), as calculated from (8.22) and (8.23), into \( \langle f, u_n \rangle \). Therefore, we have

\[
\alpha \geq F(u_n) = \langle f, u_n \rangle.
\]

**Remarks.** 1. Equations (8.23) can be considerably simplified if the set \( v_1, \ldots, v_n \) is chosen so that

\[
\langle Av_k, v_j \rangle = \delta_{kj} = \begin{cases} 
0, & k \neq j; \\
1, & k = j.
\end{cases}
\]

Then (8.23) becomes \( n \) separate equations:

\[
a_j = \langle f, v_j \rangle, \quad j = 1, \ldots, n.
\]

It therefore follows that

\[
u_n = \sum_{j=1}^{n} \langle f, v_j \rangle v_j, \quad \text{where} \quad \langle Av_k, v_j \rangle = \delta_{kj}.
\]

This suggests that the exact solution of (8.11) is

\[
u = \sum_{j=1}^{\infty} \langle f, v_j \rangle v_j,
\]

where we have used an infinite set of functions \( \{v_j\} \) suitably chosen (see below). We now try to determine under what circumstances (8.27) actually furnishes us with the solution of (8.11). To this end we introduce a new inner product in \( D_A \). This new inner product is known as the energy inner product and is denoted by \( [\ , \ ] \), where

\[
[w, z] = \langle Aw, z \rangle.
\]
Clearly \([w, z]\) is defined for each pair of elements in \(D_A\) and satisfies all the requirements [see (2.13)] on an inner product because \(A\) is a positive operator. The energy norm is defined as

\[
\langle Aw, w \rangle^{1/2} = [w, w]^{1/2},
\]

which is real and positive for \(w \neq 0\). Now let \(v_1, \ldots, v_n, \ldots\) be a set which is both orthonormal and complete in energy; that is,

\[
[v_i, v_j] = \delta_{ij},
\]

\[
\sum_{i=1}^{\infty} [w, v_i]v_i = w, \quad \text{for each } w \in D_A. \tag{8.28}
\]

Then we claim that (8.27) converges in the energy norm and represents the solution of (8.11). In fact, by (8.28), we can write

\[
u = \sum_{i=1}^{\infty} [u, v_i]v_i = \sum_{i=1}^{\infty} \langle Au, v_i \rangle v_i = \sum_{i=1}^{\infty} \langle f, v_i \rangle v_i. \tag{8.29}\]

It would appear that the entire problem of solving (8.11) has been settled since (8.29) is the required solution. Unfortunately our troubles are not quite over. The first difficulty occurs in choosing a suitable set \(\{v_i\}\). We can start with any independent set, say \(z_1, \ldots, z_n, \ldots\), and use the Gram-Schmidt orthogonalization procedure to construct a set \(\{v_i\}\) which is orthonormal in energy, but this can be an extremely laborious procedure. Further, the series in (8.29) is only known to converge in the energy norm, that is, in the sense

\[
\lim_{n \to \infty} \left[ \sum_{i=1}^{n} \langle f, v_i \rangle v_i - u, \sum_{i=1}^{n} \langle f, v_i \rangle v_i - u \right] = 0.
\]

We would like to know if the series converges in the mean to \(u\) (that is, in the ordinary norm rather than in the energy norm). It turns out that if the operator \(A\) is strongly positive (see Exercise 8.1), then convergence in energy implies convergence in the mean.

2. If \(v_1, \ldots, v_n\) are orthonormal in energy, then the Ritz-Rayleigh approximation is given by (8.26) and, since it satisfies the reciprocity principle, we have

\[
\alpha \geq F(u_n) = \langle u_n, f \rangle = \sum_{j=1}^{n} |\langle f, v_j \rangle|^2.
\]

As we add elements to set \(\{v_i\}\) we increase the right side of the inequality and obtain successively better lower bounds to \(\alpha\). In the limit, when the set \(\{v_i\}\) is complete in energy,

\[
\alpha = \sum_{j=1}^{\infty} |\langle f, v_j \rangle|^2.
\]

Again the difficulty with this formula is the orthogonalization process required in constructing a set which is orthonormal in energy.
8.5 COMPLEMENTARY VARIATIONAL PRINCIPLES

The maximum theorem applies to any linear equation of the form (8.11) when $A$ is symmetric and positive. If our interest is in estimating $\alpha$ as given by (8.12), we have

$$\alpha = F(u) \geq F(v), \quad v \in D_A.$$ 

Thus we can use the maximum principle to obtain lower bounds to $\alpha$. To find an upper bound for $\alpha$ we need a minimum principle.† If the operator $A$ on $D_A$ is a differential operator subject to boundary conditions, a complementary variational principle which is a minimum principle for $\alpha$ can be found. The original work on this subject is by Friedrichs, but here we present a simplification of these ideas due principally to Diaz.

The approach is based on three very simple inequalities. Let $\mathcal{A}$ be a complex linear space endowed with an inner product $[u, v]$ which satisfies the conditions

$$[u, v] = [v, u],$$
$$[au, v] = a[u, v], \quad \text{for any complex number } a,$$
$$[u_1 + u_2, v] = [u_1, v] + [u_2, v],$$
$$[u, u] \text{ real and nonnegative.} \quad (8.30)$$

These are just the usual conditions on an inner product, except that we allow $[u, u]$ to vanish even if $u$ is not the zero element. The following three inequalities hold in $\mathcal{A}$:

$$[u, u] \geq [u, v] + [v, u] - [v, v]. \quad (8.31)$$
$$[u, u] \geq \frac{|[u, v]|^2}{[v, v]}, \quad \text{if } [v, v] \neq 0. \quad (8.32)$$
$$[u, u] \leq [v, v], \quad \text{if } [v - u, u] = 0. \quad (8.33)$$

The first of these follows from $[u - v, u - v] \geq 0$. The second (which is Schwarz's inequality) is obtained from the first by substituting $[u, v]v/[v, v]$ for $v$. The third is derived as follows:

$$[v, v] = [u + v - u, u + v - u]$$
$$= [u, u] + [v - u, u] + [u, v - u] + [v - u, v - u]$$
$$= [u, u] + [v - u, v - u] \geq [u, u].$$

The inequality (8.33) has a simple geometric meaning, illustrated in Figure 8.1. Since $v - u$ is perpendicular to $u$, it is clear that the length $v$ is at least as

† A minimum principle for $-\alpha$ is trivially available by using $-F$ instead of $F$ as the associated functional. But since this gives an upper bound for $-\alpha$, we still have only a lower bound for $\alpha$. 

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great as the length of $u$; hence $[v, v] \geq [u, u]$. It is quite remarkable that the almost trivial inequalities (8.31), (8.32), and (8.33) generate most of the useful variational principles for linear boundary value problems.

In many applications we can use a real instead of a complex inner product. In this case the inner product $[u, v]$ is required to satisfy the conditions

\[
[u, v] = [v, u] \\
[a u, v] = a[u, v], \quad \text{for any real number } a \\
[u_1 + u_2, v] = [u_1, v] + [u_2, v] \\
[u, u] \text{ nonnegative.}
\]

The inequalities (8.31), (8.32), and (8.33) become, respectively,

\[
[u, u] \geq 2[u, v] - [v, v]. \tag{8.35}
\]

\[
[u, u] \geq \frac{[u, v]^2}{[v, v]}, \quad \text{if } [v, v] \neq 0. \tag{8.36}
\]

\[
[u, u] \leq [v, v], \quad \text{if } [v - u, u] = 0. \tag{8.37}
\]

We now illustrate the use of these inequalities in generating extremal principles for boundary value problems related to the Laplace operator in a bounded region $R$ with boundary $\sigma$. Introduce the inner product

\[
[v, w] = \int_R \nabla v \cdot \nabla w \, dx = \int_R \left( \sum_{i=1}^n \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_i} \right) dx_1 \cdots dx_n \tag{8.38}
\]

which is defined for all functions $v$ and $w$ with continuous derivatives in $\bar{R}$. We have not placed a complex conjugate on $w$ because we will be dealing with real operators and real functions. It is clear that this inner product satisfies all the properties (8.34); in particular $[v, v] \geq 0$, but $[v, v] = 0$ for any function $v$ which is constant on $R$.

First, we shall consider a problem involving Poisson's equation

\[
-\nabla^2 u = f(x), \quad x \in R; \quad u|_\sigma = 0, \tag{8.39}
\]

where $f(x)$ is a real, continuous function on $\bar{R}$. In addition to (or instead of) finding $u$, we would like to estimate the quantity

\[
\alpha = \int_R \nabla u \cdot \nabla u \, dx = \int_R |\nabla u|^2 \, dx = \int_R f(x)u(x) \, dx, \tag{8.40}
\]
where the last equality is obtained by using Green's theorem and (8.39). The system (8.39) occurs in the torsion of a cylinder whose cross section is the plane region $R$. The stress function $u(x) = u(x_1, x_2)$ then satisfies system (8.39) with $f(x) = 2$. The constant of proportionality which relates the applied twisting moment and the angle of twist per unit length of the cylinder is known as the torsional rigidity and is denoted by $D$. It can be shown (see, for instance, I. S. Sokolnikoff, *Mathematical Theory of Elasticity*, 2nd ed., McGraw-Hill, New York, 1956) that

$$D = \int_R 2u(x)dx = \int_R f(x)u(x)dx = \int_R |\text{grad } u|^2 \, dx. \quad (8.41)$$

Here we have set the shear modulus equal to 1 for convenience. Thus $D$ coincides with the quantity $\alpha$ [see (8.40) with $f(x) = 2$]. The quantity $D$ is an excellent measure of the resistance of the cylinder to the applied twisting moment and we are often content to determine $D$ rather than to find $u(x)$.

Returning to the general case (8.39) with arbitrary $f(x)$, we develop variational principles to characterize $u(x)$ and $\alpha$.

(a) Using (8.37), we have

$$\alpha = \int_R |\text{grad } u|^2 \, dx \leq \int_R |\text{grad } v|^2 \, dx \quad (8.42)$$

if

$$\int_R \text{grad}(v - u) \cdot \text{grad } u \, dx = 0. \quad (8.43)$$

The inequality (8.42) will provide a useful estimate of $\alpha$ only if we can calculate $\int_R |\text{grad } v|^2 \, dx$, that is, only if we can explicitly characterize functions satisfying (8.43). This can be easily done even though we cannot find $u$ explicitly! Indeed the condition (8.43) reduces to

$$0 = -\int_R u\nabla^2(v - u)dx + \int_\sigma u \frac{\partial}{\partial n} (v - u)dS,$$

and since $u$ vanishes on $\sigma$,

$$0 = \int_R u\nabla^2(v - u)dx.$$  

Thus a function $v$ will satisfy (8.43) if $\nabla^2 v = \nabla^2 u$, or $-\nabla^2 v = f$. We can therefore write

$$\alpha \leq \int_R |\text{grad } v|^2 \, dx, \quad \text{for every } v \text{ satisfying } -\nabla^2 v = f.$$

Further, we easily see that for the admissible function $v = u$, the right side of the inequality is exactly $\alpha$. Therefore,

$$\min_{v \in \mathbb{V} \quad -\nabla^2 v = f} \int_R |\text{grad } v|^2 = \alpha; \quad \text{minimum occurs for } v = u. \quad (8.44)$$
(b) To obtain a maximum principle we may use either (8.35) or (8.36). Starting with (8.35), we have

\[ \alpha \geq 2[u, v] - [v, v] = 2 \int_R \nabla u \cdot \nabla v \, dx - \int_R |\nabla v|^2 \, dx. \]

The right side can be calculated without knowing \( u \). In fact,

\[ \int_R \nabla u \cdot \nabla v \, dx = -\int_R v \nabla^2 u \, dx + \int_{\partial R} v \frac{\partial u}{\partial n} \, dS, \]

so that, since \(-\nabla^2 u = f\), we have

\[ \int_R \nabla u \cdot \nabla v \, dx = \int_R f v \, dx, \quad \text{if} \ v|_{\partial R} = 0. \]

Thus

\[ \alpha \geq 2 \int_R f v \, dx - \int_R |\nabla v|^2 \, dx, \quad \text{if} \ v|_{\partial R} = 0. \]

We observe that the maximum on the right is attained for \( v = u \), so that

\[ \alpha = \max_{v|_{\partial R} = 0} \left[ 2 \int_R f v \, dx - \int_R |\nabla v|^2 \, dx \right], \tag{8.45} \]

and the maximum occurs for \( v = u \). The principle (8.45) is exactly the maximum theorem (8.14) for the present case. The maximum principle (8.45) can be slightly altered by going back to the Schwarz inequality (8.36). We then find

\[ \alpha = \max_{v|_{\partial R} = 0, \ v \text{ constant}} \frac{\left[ \int_R f v \, dx \right]^2}{\int_R |\nabla v|^2 \, dx}, \tag{8.46} \]

and the maximum occurs for \( v = cu \), where \( c \) is any nonzero constant. This is just the present version of (8.18).

As an example of the use of these principles to obtain estimates on \( \alpha \), let us turn to the problem of torsional rigidity. Then \( \alpha = D \) and \( f(x) = 2 \). We have

\[ \frac{\left[ \int_R f w \, dx \right]^2}{\int_R |\nabla w|^2 \, dx} \leq D \leq \int_R |\nabla v|^2 \, dx, \]

where \( v \) is any function such that \(-\nabla^2 v = 2\) and \( w \) is any nonconstant function such that \( w|_{\partial R} = 0 \).
Now the function \( v_0 = -(x_1^2 + x_2^2)/2 \) clearly satisfies \(-\nabla^2 v_0 = 2\), for an arbitrary choice of the origin. Therefore,

\[
D \leq \int_R (x_1^2 + x_2^2) \, dx_1 \, dx_2.
\]

The right side is as small as possible if we choose for the origin the center of gravity. Then the integral is just the polar moment of inertia \( I \). Hence we have the interesting, if somewhat crude, inequality

\[
D \leq I. \tag{8.47}
\]

If we add to \( v_0 \) any harmonic function, then we have a function \( v \) which still satisfies \(-\nabla^2 v = 2\). Choosing

\[
v = v_0 + \frac{A}{2} (x_1^2 - x_2^2),
\]

then

\[
D \leq \int_R [(A - 1)^2 x_1^2 + (A + 1)^2 x_2^2] \, dx_1 \, dx_2.
\]

Selecting \( A \) to minimize the right side we obtain a more precise inequality than (8.47),

\[
D \leq \frac{4I_1 I_2}{I}, \tag{8.48}
\]

where \( I_1 \) and \( I_2 \) are the moments of inertia of the cross section about the \( x_1 \) and \( x_2 \) axes, respectively. The Ritz-Rayleigh procedure can be applied to any of these principles, so that bounds for \( D \) may be obtained as precisely as desired.

Next we consider the Dirichlet problem for Laplace's equation in a bounded region \( R \) with boundary \( \sigma \). We wish to find the function \( u(x) \) which satisfies

\[
\nabla^2 u = 0, \quad x \in R; \quad u \mid_\sigma = f, \tag{8.49}
\]

where \( f \) is a real continuous function. In addition to or in place of finding \( u \), we would like to estimate the quantity

\[
\alpha = \int_R \nabla u \cdot \nabla u \, dx = \int_R |\nabla u|^2 \, dx = \int_\sigma f \frac{\partial u}{\partial n} \, dS, \tag{8.50}
\]

the last equality being obtained by using Green's theorem and the fact that \( u \) is harmonic in \( R \).

As we have seen in Chapter 6, (8.49) has one and only one solution and we shall therefore not be concerned here with questions of existence.

We introduce again the inner product (8.38). We wish to estimate \( \alpha \), which can be written as \([u, u]\). From (8.35), we have for any \( v \),

\[
[u, u] \geq 2[u, v] - [v, v] = 2 \int_R \nabla u \cdot \nabla v \, dx - \int_R |\nabla v|^2 \, dx.
\]
The right side can be calculated even if \( u \) is not known! In fact,

\[
\int_R \nabla u \cdot \nabla v \, dx = -\int_R u \nabla^2 v \, dx + \int_\partial R \frac{\partial v}{\partial n} \, dS
\]

\[
= -\int_R u \nabla^2 v \, dx + \int_\partial R f \frac{\partial v}{\partial n} \, dS.
\]

If we restrict ourselves to functions \( v \) which are harmonic (\( \nabla^2 v = 0 \)), then

\[
\int_R \nabla u \cdot \nabla v \, dx = \int_\partial R f \frac{\partial v}{\partial n} \, dS.
\]

Thus we have

\[
\alpha = [u, u] \geq 2 \int_\partial R f \frac{\partial v}{\partial n} \, dS - \int_R |\nabla v|^2 \, dx,
\]

for any \( v \) such that \( \nabla^2 v = 0 \). Observe also that if we take the maximum value of the right side over all harmonic functions \( v \) we get just \( \alpha \); indeed, the function \( u \) itself is harmonic, and on substituting \( u \) for \( v \), the right side reduces to \( \alpha \) by (8.50). Therefore, we have the following maximum principle for \( \alpha \):

\[
\alpha = \max_{\nu \neq \nabla^2 \nu = 0} \left[ 2 \int_\partial R f \frac{\partial v}{\partial n} \, dS - \int_R |\nabla v|^2 \, dx \right]. \tag{8.51}
\]

Using instead Schwarz’s inequality (8.36), we obtain

\[
\alpha = \max_{\nu \neq \nabla^2 \nu = 0} \left[ \int_\partial R \frac{\partial v}{\partial n} \, dS \right]^2 \int_R |\nabla v|^2 \, dx. \tag{8.52}
\]

Here the maximum occurs not only for \( v = u \) but for any function \( v \) proportional to \( u \) (that is, of the form \( cu \), where \( c \) is a nonzero constant).

A minimum principle for \( \alpha \) is a consequence of (8.37). We have

\[
\alpha = [u, u] \leq \int_R |\nabla v|^2 \, dx, \quad \text{if} \int_R \nabla(v-u) \cdot \nabla u \, dx = 0.
\]

The last condition can be written

\[
0 = -\int_R (v-u) \nabla^2 u \, dx + \int_\partial R (v-u) \frac{\partial u}{\partial n} \, dS,
\]

and, since \( \nabla^2 u = 0 \), this equation is satisfied if \( v \mid_\partial = f \). Therefore,

\[
\alpha \leq \int_R |\nabla v|^2 \, dx, \quad \text{if} \ v \mid_\partial = f.
\]
Now the functional on the right side is exactly $\alpha$ if we substitute the admissible function $v = u$. Therefore,

\[
\alpha = \min_{v|_\sigma = f} \int_R |\text{grad } v|^2 \, dx,
\]

the minimum value occurring for $v = u$.

### 8.6 Capacity Problem

Let $R_e$ be the exterior of a bounded region $R_i$ in three dimensions, and let the boundary of $R_i$ be $\sigma$. Consider the following problem in electrostatics: To find the potential $u(x)$ in $R_e$ if $u|_\sigma = 1$ and the potential at infinity vanishes. Thus $u(x)$ satisfies the exterior Dirichlet problem (6.139)

\[
\nabla^2 u = 0, \quad x \text{ in } R_e; \quad u|_\sigma = 1,
\]

\[
|x|u \text{ bounded at } \infty, \quad |x|^2 |\text{grad } u| \text{ bounded at } \infty.
\]

Although we would of course be extremely pleased to be able to solve for $u(x)$, we are willing to settle for an estimate of the capacity $C$ which is defined as

\[
C = -\int_\sigma \frac{\partial u}{\partial n} \, dS,
\]

where $n$ is the normal to $\sigma$ outward from $R_i$. If we multiply (8.54) by $u$ and integrate over $R_e$, we find

\[
0 = \int_{R_e} u\nabla^2 u \, dx = -\int_{R_e} |\text{grad } u|^2 \, dx + \int_{\sigma_\infty} u \frac{\partial u}{\partial n} \, dS - \int_{\sigma} u \frac{\partial u}{\partial n} \, dS.
\]

The second integral on the right vanishes by the behavior of $u$ and grad $u$ at infinity. Therefore, since $u|_\sigma = 1$,

\[
C = -\int_{\sigma} \frac{\partial u}{\partial n} \, dS = \int_{R_e} |\text{grad } u|^2 \, dx.
\]

This last equality shows that the capacity of the conductor is equal to the electrostatic energy stored in the field.

By methods similar to those in the preceding example, we obtain variational principles for $C$. Let $\mathcal{M}$ be the set of functions $v$ which have continuous second derivatives in $R_e$ and such that $|x|v$ and $|x|^2 |\text{grad } v|$ are bounded at infinity. For any two functions $v$ and $w$ in $\mathcal{M}$, we define the inner product

\[
[v, w] = \int_{R_e} \text{grad } v \cdot \text{grad } w \, dx.
\]

Then

\[
C = [u, u].
\]
and, by (8.37),
\[ C = [u, u] \leq [v, v], \quad \text{if} \ [v - u, u] = 0. \]

Now
\[ [v - u, u] = \int_{R \varepsilon} \text{grad} \ u \cdot \text{grad} (v - u) \, dx = - \int_{\partial} (v - u) \frac{\partial u}{\partial n} \, dS, \]
so that \([v - u, u] = 0, \text{if } v|_{\partial} = 1\). Therefore,
\[ C = \min_{v \in \mathcal{M}, \ v|_{\partial} = 1} \int_{R \varepsilon} |\text{grad} \ v|^2 \, dx. \quad (8.57) \]

Applying (8.35), we have
\[ C \geq 2 \int_{R \varepsilon} (\text{grad} \ u \cdot \text{grad} \ v) \, dx - \int_{R \varepsilon} |\text{grad} \ v|^2 \, dx. \]

Since
\[ \int_{R \varepsilon} \text{grad} \ u \cdot \text{grad} \ v \, dx = - \int_{R \varepsilon} u \nabla^2 v \, dx - \int_{\partial} u \frac{\partial v}{\partial n} \, dS, \]
we have
\[ C = \max_{u \in \mathcal{M}, \ \nabla^2 u = 0} \left[ -2 \int_{\partial} \frac{\partial v}{\partial n} \, dS - \int_{R \varepsilon} |\text{grad} \ v|^2 \, dx \right]. \quad (8.58) \]

We can obtain a scale-independent result by using (8.36). This yields
\[ C = \max_{v \in \mathcal{M}, \ \nabla^2 v = 0} \frac{\left( \int_{\partial} \frac{\partial v}{\partial n} \, dS \right)^2}{\int_{R \varepsilon} |\text{grad} \ v|^2 \, dx}. \quad (8.59) \]

In (8.57) and (8.58) the extremal value is attained for \( v = u \) and in (8.59) for \( v = cu \), where \( c \) is a nonzero constant.

The capacity problem can also be formulated as an integral equation on \( \sigma \). In fact, by (6.140), we know that the charge density \( \rho(x) \) on \( \sigma \) satisfies the integral equation
\[ 1 = \int_{\sigma} \frac{1}{4\pi |x - \xi|} \rho(\xi) \, dS_\xi, \quad x \text{ on } \sigma. \quad (8.60) \]

Once this equation is solved for \( \rho \), the electrostatic potential \( u \) is found from
\[ u(x) = \int_{\sigma} \frac{1}{4\pi |x - \xi|} \rho(\xi) \, dS_\xi, \quad x \text{ in } R_\varepsilon. \quad (8.61) \]

Rather than solve (8.60) we try to estimate the capacity \( C \). Since
\[ C = \int_{\sigma} \rho(x) \, dS_x, \]

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we have from (8.60), by multiplying by \( \rho(x) \) and integrating over \( \sigma \),

\[
C = \int_\sigma \rho(x) dS_x = \int_\sigma \int dS_x dS_\xi \frac{\rho(x) \rho(\xi)}{4\pi |x - \xi|}.
\] (8.62)

It can be proved that the integral operator generated by the kernel \( 1/4\pi |x - \xi| \) on \( \sigma \) is positive. Introducing the inner product

\[
\{v, w\} = \int_\sigma \int dS_x dS_\xi \frac{v(x) w(\xi)}{4\pi |x - \xi|},
\] (8.63)
defined for continuous functions on \( \sigma \), we conclude from (8.35) that

\[
C = \{\rho, \rho\} = \max_v \left[ 2 \int_\sigma v(x) dS_x - \int_\sigma \int dS_x dS_\xi \frac{v(x) v(\xi)}{4\pi |x - \xi|} \right].
\] (8.64)

From (8.36), we have

\[
C = \max_v \frac{\left[ \int_\sigma v(x) dS_x \right]^2}{\int_\sigma \int dS_x dS_\xi \frac{v(x) v(\xi)}{4\pi |x - \xi|}}.
\] (8.65)

If we attempt to use (8.37) to get a minimum principle, we face the problem of characterizing functions \( v(x) \) on \( \sigma \) for which \( \{v - \rho, \rho\} = 0 \). This leads to

\[
\int_\sigma v(x) dS_x = \int_\sigma \rho(x) dS_x,
\]

but since the quantity on the right is just the unknown capacity we are trying to find, the method fails.

### 8.7 Natural Boundary Conditions

We return to the general problem (8.11),

\[
Au = f, \quad u \in D_A,
\] (8.66)

where \( A \) is a positive symmetric operator on a linear manifold \( D_A \) dense in the Hilbert space \( \mathcal{A} \). When dealing with boundary value problems \( A \) will be a differential operator on a domain \( D_A \) characterized by the homogeneous boundary conditions satisfied by \( u \). Assuming that (8.66) has a solution, then \( u \) is the function which maximizes, among all functions \( v \) in \( D_A \), the functional

\[
F(v) = \langle f, v \rangle + \langle v, f \rangle - \langle Av, v \rangle.
\] (8.67)

As we have seen in examples the form \( \langle Av, v \rangle \) usually reduces, through integration by parts and use of the boundary conditions, to an expression involving only derivatives of lower order than those appearing in (8.66). This suggests that the requirement that admissible functions should belong
to $D_A$ might be relaxed at least insofar as differentiability properties are concerned. Further, it will sometimes be possible to seek the maximum of (8.67) among functions which do not satisfy all the boundary conditions in $D_A$ and still find that the maximum occurs for the function $u$ which is the solution of (8.66). Any boundary condition in (8.66) which need not be imposed on the set of admissible functions in (8.67) is said to be a natural boundary condition. All other boundary conditions are said to be essential.

To illustrate these ideas, we consider two simple examples which will be studied together. The first boundary value problem is

$$-u''_1 = f, \quad 0 < x < l; \quad u_1(0) = u_1(l) = 0,$$  \hspace{1cm} (8.68)

and the second is

$$-u''_2 = f, \quad 0 < x < l; \quad u_2(0) = u_2'(l) = 0.$$  \hspace{1cm} (8.69)

The only difference between these problems is the boundary condition at $x = l$. We shall assume (although this is not necessary) that $f$ is a continuous real function. As is known from the theory of differential equations (see, for instance, Chapter 1) each system has one and only one solution. Let $A$ be the operator $-d^2/dx^2$ defined on the domain $D_A$ of real functions having a continuous second derivative and satisfying the boundary conditions in (8.68). Similarly, $B$ is the operator $-d^2/dx^2$ defined on the domain $D_B$ of real functions having a continuous second derivative and satisfying the boundary conditions in (8.69). The operators $A$ and $B$ are easily seen to be symmetric and positive on their respective domains. Therefore by our maximum theorem we can characterize $u_1$ from the maximum principle

$$F(u_1) = \max_{v \in D_A} F(v),$$  \hspace{1cm} (8.70)

where

$$F(v) = 2\langle f, v \rangle - \langle Av, v \rangle = 2 \int_0^l f v \, dx + \int_0^l vv'' \, dx.$$  \hspace{1cm} (8.71)

Since $v$ is in $D_A$, we have

$$- \int_0^l vv'' \, dx = \int_0^l (v')^2 \, dx.$$

Thus $u_1$ is also characterized by

$$F(u_1) = G(u_1) = \max_{v \in D_A} G(v); \quad \text{maximum occurs for } v = u_1,$$  \hspace{1cm} (8.72)

where

$$G(v) = 2 \int_0^l f v \, dx - \int_0^l (v')^2 \, dx.$$  \hspace{1cm} (8.73)

Similarly, we find that $u_2$ is characterized by

$$F(u_2) = G(u_2) = \max_{v \in D_B} G(v); \quad \text{maximum occurs for } v = u_2.$$  \hspace{1cm} (8.74)
We observe that $F(v) = G(v)$ whenever $v$ is in $D_A$ or $D_B$ but not necessarily otherwise.

Let us now try to enlarge the class of admissible functions which can be used in the variational principle (8.72). We let $v = u_1 + h$, where for the time being we require only that $h$ have a piecewise continuous first derivative. Then we can calculate $G(u_1 + h)$:

$$ G(u_1 + h) = G(u_1) + 2 \int_0^l fh\, dx - 2 \int_0^l u_1' h'\, dx - \int_0^l (h')^2\, dx, $$

or

$$ G(u_1 + h) - G(u_1) = 2 \int_0^l fh\, dx + 2 \int_0^l u_1' h\, dx - \int_0^l (h')^2\, dx - 2u_1' h\big|_0^l. $$

Since $f + u_1'' = 0$, we find

$$ G(u_1 + h) - G(u_1) = - \int_0^l (h')^2\, dx - 2u_1' (l) h(l) + 2u_1' (0) h(0). \quad (8.75) $$

The right side will be negative if $h(0) = h(l) = 0$ and $h(x)$ is not identically 0. Thus if we introduce the set $C_1$ of functions having a piecewise continuous first derivative, we have

$$ G(u_1) = \max_{v \in C_1} G(v); \quad \text{maximum occurs for } v = u_1. \quad (8.76) $$

The principle (8.76) is only a little stronger than (8.72). We can now obtain a lower bound to $G(u_1)$ by using functions which do not have a second derivative. Sometimes in applying the Ritz-Rayleigh procedure one wants to use continuous, piecewise linear, functions as approximating functions; we have shown that this procedure is acceptable.

We now turn to the maximum principle (8.74) associated with (8.69). As before we find, if $h$ has a piecewise continuous first derivative,

$$ G(u_2 + h) - G(u_2) = - \int_0^l (h')^2\, dx - 2u_2' (l) h(l) + 2u_2' (0) h(0). $$

Since $u_2' (l) = 0$, we see that the right side will be negative if $h(0) = 0$ and $h(x)$ is not identically zero. Therefore,

$$ G(u_2) = \max_{v \in C_1} G(v); \quad \text{maximum occurs for } v = u_2. \quad (8.77) $$

The second boundary condition in (8.69) need not be imposed on the admissible functions in the variational principle and is therefore a natural boundary condition; the boundary condition at the left end is essential. There are two advantages to the formulation (8.77). First, in applying the Ritz-Rayleigh procedure, we need not require that the functions $v_1, \ldots, v_n$ satisfy the condition $v_i' (l) = 0$; this can considerably simplify the choice of the set.
\{v_i\} and the resulting calculations but, of course, it must be realized that if the set \{v_i\} satisfies \(v_i(l) = 0\), the convergence of the procedure is more rapid. Second, we can now compare \(G(u_2)\) and \(G(u_1)\). In fact, we have the same functional but one set of admissible functions contains the other. Obviously the maximum over the larger set is greater than or equal to the maximum over the smaller set. Thus

\[
G(u_2) \geq G(u_1). \tag{8.78}
\]

The physical interpretation of this inequality is of considerable importance and clarifies the notion of natural boundary condition. We may think of \(u_1\) as the deflection of a string with fixed ends, whereas \(u_2\) is the deflection of a string one of whose ends is fixed and the other free. It is clear that under the same transverse pressure \(f(x)\), the string with fixed ends will have a potential energy greater than the other. Now as we saw in (8.3) the potential energy \(W\) is \(-\frac{1}{2}G\). Thus, on physical grounds, we expect

\[-\frac{1}{2}G(u_1) \geq -\frac{1}{2}G(u_2),\]

which is just (8.78).

### 8.8 INDEFINITE AND NONSYMMETRIC OPERATORS

If \(A\) is symmetric but indefinite (that is, \(\langle Av, v \rangle\) can be positive or negative, depending on \(v\)) or if \(A\) is not symmetric the extremal principles previously derived do not hold in the sense of minimum or maximum principles. We can still salvage something useful by introducing stationary principles.

**Definition.** A functional \(F(v)\) is said to be stationary at \(u\) in the class of admissible functions \(M\), if

\[
\left. \frac{d}{d\epsilon} F(u + \epsilon \eta) \right|_{\epsilon = 0} = 0, \quad \text{for each } \eta \text{ in } M. \tag{8.79}
\]

Observe that if \(F\) has either a maximum or a minimum at \(u\), then it is stationary at \(u\). But of course (8.79) can hold even if \(F\) has neither a maximum or minimum at \(u\); in fact, (8.79) only implies that, for small \(\epsilon, F(u + \epsilon \eta) - F(u)\) is of order \(\epsilon^2\). This is, of course, similar to what happens at a stationary point for a function of a real variable.

Now let \(A\) be a symmetric linear operator defined on the domain \(D_A\). We are interested in characterizing the solution \(u\) of

\[
Au = f, \quad u \in D_A. \tag{8.80}
\]

At the same time we would like to estimate the numerical quantity \(\alpha = \langle f, u \rangle\). We observe from (8.80) that

\[
\alpha = \langle f, u \rangle = \langle Au, u \rangle = \langle u, Au \rangle = \langle u, f \rangle, \tag{8.81}
\]
so that $\alpha$ is real, but of course we do not know whether $\alpha$ is positive or negative. We shall assume that (8.80) has one and only one solution. Then we have

**Theorem (Stationary Principle).** The one and only function which makes the functional

$$F(v) = \langle f, v \rangle + \langle v, f \rangle - \langle Av, v \rangle \quad (8.82)$$

stationary in the class $D_A$ of admissible functions is the function $v = u$, where $u$ is the solution of (8.80).

**Proof.** (a) We want to prove that $u$, the solution of (8.80), makes $F$ stationary, that is,

$$\frac{d}{d\varepsilon} F(u + \varepsilon \eta) \bigg|_{\varepsilon = 0} = 0, \text{ for each } \eta \text{ in } D_A.$$

We have, from (8.82),

$$\frac{d}{d\varepsilon} F(u + \varepsilon \eta) \bigg|_{\varepsilon = 0} = \langle f, \eta \rangle + \langle \eta, f \rangle - \langle A\eta, u \rangle - \langle Au, \eta \rangle$$

$$= \langle f - Au, \eta \rangle + \langle \eta, f - Au \rangle = 0.$$

(b) Suppose $u$ makes $F$ stationary in $D_A$; we want to show that $u$ is the solution of (8.80). For each $\eta$ in $D_A$, we have

$$0 = \frac{d}{d\varepsilon} F(u + \varepsilon \eta) \bigg|_{\varepsilon = 0} = \langle f - Au, \eta \rangle + \langle \eta, f - Au \rangle.$$

Since $D_A$ is a linear manifold, $in$ is also in $D_A$, so that

$$0 = - \langle f - Au, \eta \rangle + \langle \eta, f - Au \rangle.$$

Adding to the previous equation we find

$$\langle \eta, f - Au \rangle = 0, \quad \text{for all } \eta \text{ in } D_A.$$

Since $D_A$ is dense in $\mathcal{A}$, this implies $f - Au = 0$.

We also note that the stationary value of $F$ is $F(u)$, which, according to (8.81), is just $\alpha$. Of course, in this case we cannot guarantee whether an approximation to $u$, when substituted in (8.82), will yield an upper or lower bound to $\alpha$.

When dealing with operators $A$ which are not symmetric, we will have to introduce functionals which depend on two functions.

**Definition.** A functional $F(w, z)$ is said to be stationary at $(u, v)$ in the admissible class $(M_1, M_2)$ if

$$\frac{\partial}{\partial \varepsilon} F(u + \varepsilon \eta, v) \bigg|_{\varepsilon = 0} = \frac{\partial}{\partial \gamma} F(u, v + \gamma \zeta) \bigg|_{\gamma = 0} = 0, \quad (8.83)$$

for each $\eta$ in $M_1$ and each $\zeta$ in $M_2$. 

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We shall be interested in the equation

\[ Au = f, \quad u \in D_A \]  \hspace{1cm} (8.84)

and in the numerical quantity \( \alpha = \langle u, g \rangle \), where \( g \) is an element that may be different from \( f \). We shall need to introduce the auxiliary problem

\[ A^*v = g, \quad v \in D_{A^*}, \]  \hspace{1cm} (8.85)

where \( A^* \) is the adjoint of \( A \) and \( D_{A^*} \) is the domain of \( A^* \). We assume that (8.84) as well as (8.85) have unique solutions. From (8.84) and (8.85) we observe that

\[ \alpha = \langle u, g \rangle = \langle u, A^*v \rangle = \langle Au, v \rangle = \langle f, v \rangle. \]  \hspace{1cm} (8.86)

**Theorem (Stationary Principle).** The one and only pair of functions which makes the functional

\[ F(w, z) = \langle w, g \rangle + \langle f, z \rangle - \langle Aw, z \rangle \]  \hspace{1cm} (8.87)

stationary in the class \( (D_A, D_{A^*}) \) is the pair \( w = u, z = v \). The stationary value of \( F \) is \( F(u, v) = \alpha \).

**Proof.** (a) We have

\[ \frac{\partial}{\partial \epsilon} F(u + \epsilon \eta, v) \bigg|_{\epsilon = 0} = \langle \eta, g \rangle - \langle A\eta, v \rangle = \langle \eta, g - A^*v \rangle = 0, \]

and

\[ \frac{\partial}{\partial \gamma} F(u, v + \gamma \zeta) \bigg|_{\gamma = 0} = \langle f, \zeta \rangle - \langle Au, \zeta \rangle = \langle f - Au, \zeta \rangle = 0. \]

(b) Let \( (u, v) \) be a pair of functions which makes \( F \) stationary in the class \( (D_A, D_{A^*}) \). Then, we have

\[ \langle \eta, g - A^*v \rangle = 0 \quad \text{for each} \ \eta \ \text{in} \ \mathcal{A}, \]

\[ \langle f - Au, \zeta \rangle = 0 \quad \text{for each} \ \zeta \ \text{in} \ \mathcal{A}^*. \]

Since \( D_A \) and \( D_{A^*} \) are dense in \( \mathcal{A} \), it follows that \( f - Au = 0 \) and \( g - A^*v = 0 \).

**Schwinger-Levine principle.** By the same arguments used for positive operators, it can be shown that the stationary principle just derived can be rewritten in a scale-independent form using the functional

\[ R(w, z) = \frac{\langle w, g \rangle \langle f, z \rangle}{\langle Aw, z \rangle}. \]  \hspace{1cm} (8.88)

The only pairs of functions which make \( R \) stationary are \( w = c_1u, z = c_2v \), where \( c_1 \) and \( c_2 \) are arbitrary constants. Again the stationary value of \( R \), that is \( R(c_1u, c_2v) \), is just equal to \( \alpha \).
8.9 OTHER METHODS FOR UPPER BOUNDS TO FUNCTIONALS ASSOCIATED WITH POSITIVE OPERATORS

Let \( A \) be a real symmetric positive operator on a Hilbert space and let \( P \) be the projection operator on a subspace \( \mathcal{M} \) of \( \mathcal{A} \). In the Ritz-Rayleigh procedure described in Section 8.4, one obtains an approximate solution to \( Au = f \) by considering instead the equation \( PAu = Pf \) with \( u \) in \( \mathcal{M} \); usually \( \mathcal{M} \) is chosen as a finite-dimensional subspace of \( \mathcal{A} \), so that the equation \( PAu = Pf \) reduces to a set of simultaneous algebraic equations considerably simpler than the equation \( Au = f \). Here we shall take a different view; we shall assume that we know how to solve the equation \( Au = f \) but not the equation \( PAu = Pf \). We are then interested in finding an approximate solution of this latter equation and at the same time obtain an upper bound to the associated functional \( \langle Pf, u \rangle \). Situations of this type occur frequently for integral equations with \( P \) a projection on an infinite-dimensional subspace of \( \mathcal{A} \) (see the example following the development of the theory).

We wish to find an approximate solution of

\[
P Au = Pf, \quad u = Pu, \quad (8.89)
\]

and to obtain an upper bound to

\[
\alpha = \langle Pf, u \rangle = \langle PAu, u \rangle = \langle Au, u \rangle. \quad (8.90)
\]

From (8.89) we also have

\[
Au = Pf + g, \quad (8.91)
\]

where \( g \) is unknown but \( Pg = 0 \). Conceptually we can use (8.91) to find \( u \). Indeed since \( A^{-1} \) is known, we can solve (8.91) for any choice of \( g \); it remains only to find the function \( g \) that will give a solution for which \( Pu = u \). This point of view can be employed to derive the results below but we prefer to adopt a different approach.

In \( \mathcal{A} \) we introduce the new inner product

\[
[w, z] = \langle w, Az \rangle,
\]

in terms of which

\[
\alpha = [u, u].
\]

From (8.33) we have

\[
\alpha \leq [v, v], \quad \text{ if } [v - u, u] = 0,
\]

so that we need to characterize elements \( v \) in \( \mathcal{A} \) for which \( [v - u, u] = 0 \). This equation is equivalent to \( \langle PAv - PAu, u \rangle = 0 \) which is satisfied if \( PAv = Pf \). One should observe that this equation is not equivalent to (8.89) since \( v \) is not necessarily equal to \( Pu \). Thus

\[
\alpha \leq [v, v] = \langle Av, v \rangle, \quad \text{for all } v \in PAv = Pf, \quad (8.92)
\]

and, by choosing \( v = u \), the inequality becomes an equality.
We now rewrite (8.92) with a view toward setting up an approximation scheme. Let \( v_0 \) be an arbitrary fixed function such that \( PAv_0 = Pf \); that is,
\[
Av_0 = Pf + g_0, \quad \text{with } Pg_0 = 0. \tag{8.93}
\]
Since \( A^{-1} \) is known, this equation can be solved for any \( g_0 \), but convenience or physical considerations often dictate a particularly useful choice of \( g_0 \). We define
\[
\alpha_0 = \langle Av_0, v_0 \rangle = \langle Pf + g_0, v_0 \rangle.
\]
Any function \( v \) for which \( PAv = Pf \) can be written \( v = v_0 + w \), with \( Aw = q \) and \( Pq = 0 \). Substituting in (8.92), we find
\[
\alpha \leq \alpha_0 + 2\langle v_0, q \rangle + \langle q, A^{-1}q \rangle, \quad Pq = 0. \tag{8.94}
\]
The right side of this inequality reduces to \( \alpha \) if \( q \) is chosen equal to \( g - g_0 \), where \( g \) and \( g_0 \) are defined from (8.91) and (8.93), respectively. Thus the function \( g - g_0 \) is characterized by an extremal principle and we can use a Ritz-Rayleigh type of procedure to find an approximation to \( g - g_0 \). Let \( \Psi_1, \ldots, \Psi_n \) be real independent functions with the property \( P\Psi_k = 0, \ k = 1, \ldots, n \). We substitute for \( q \) in (8.94) the real linear combination
\[
\sum_{k=1}^{n} c_k \Psi_k,
\]
and adjust the coefficients so as to minimize the right side of (8.94). This leads to the set of algebraic equations
\[
\langle v_0, \Psi_k \rangle = -\sum_{j=1}^{n} c_j \langle A^{-1}\Psi_j, \Psi_k \rangle, \quad k = 1, \ldots, n. \tag{8.95}
\]
If these equations are solved for \( c_1, \ldots, c_n \), we have \( q^* = \sum_{k=1}^{n} c_k \Psi_k \) as an approximation to \( g - g_0 \). Since \( g - g_0 = A(u - v_0) \), we find
\[
u = v_0 + A^{-1}(g - g_0),
\]
and the corresponding approximation \( u^* \) to \( u \) is
\[
u^* = v_0 + \sum_{k=1}^{n} c_k A^{-1}\Psi_k. \tag{8.96}
\]
Our principal goal is not to approximate \( u \) but rather to find an upper bound to \( \alpha \) from (8.94). This is achieved by substituting \( q^* = \sum_{k=1}^{n} c_k \Psi_k \) into (8.94). We can simplify matters a little by observing that \( q^* \) satisfies the reciprocity principle
\[
\langle v_0, q^* \rangle = -\langle A^{-1}q^*, q^* \rangle,
\]
which follows directly from (8.95). Thus the inequality (8.94) becomes
\[
\alpha \leq \alpha_0 - \langle A^{-1}q^*, q^* \rangle = \alpha_0 + \langle v_0, q^* \rangle.
\]
If a single function $\Psi$ is used in (8.95), we find the corresponding coefficient

$$c = -\frac{\langle v_0, \Psi \rangle}{\langle A^{-1} \Psi, \Psi \rangle},$$

$$\alpha \leq \alpha_0 - \frac{\langle v_0, \Psi \rangle^2}{\langle A^{-1} \Psi, \Psi \rangle}, \quad P\Psi = 0. \quad (8.97)$$

Of course, the usual lower bound to $\alpha$ is available by applying the Ritz-Rayleigh method directly to the operator $A' = PA$.

**EXAMPLE**

Let $\mathcal{A}$ be the space of real $L_2$ functions on the interval $(-\pi, \pi)$ and let the operator $A$ be defined as

$$Az = Kz + z = \int_{-\pi}^{\pi} k(x - \xi)z(\xi)d\xi + z(x), \quad -\pi < x < \pi. \quad (8.98)$$

For this operator to define a function in $(-\pi, \pi)$ we must know $k(x)$ for values of the argument between $-2\pi$ and $2\pi$. We shall assume that $k(x)$ is a real, even function with period $2\pi$ and that the integral operator $k$ it generates is a nonnegative Hilbert-Schmidt operator. Then the operator $A$ is clearly positive. To solve the equation

$$Az = f, \quad -\pi < x < \pi, \quad (8.99)$$

we use a Fourier expansion. Letting $k_n$, $f_n$, and $z_n$ stand for the Fourier coefficients of $k(x)$, $f(x)$, and $z(x)$, respectively, we find

$$z(x) = \sum_{n=-\infty}^{\infty} z_ne^{inx} = \sum_{n=-\infty}^{\infty} \frac{f_n}{1 + 2\pi k_n} e^{inx}.$$

On the other hand, we cannot use this method to solve the integral equation

$$\int_{0}^{\pi} k(x - \xi)u(\xi)d\xi + u(x) = f(x), \quad 0 < x < \pi. \quad (8.100)$$

The reason for the difficulty is that $k$ has a period which is inappropriate for the interval in which we want to solve the integral equation. We can write (8.100) in the form (8.89) by introducing the projection operator $P$ defined as

$$Pw = \begin{cases} w(x), & 0 < x < \pi; \\ 0, & -\pi < x < 0. \end{cases} \quad (8.101)$$

The operator $P$ projects on the subspace $\mathcal{M}$ of $\mathcal{A}$ consisting of all functions which vanish identically for $-\pi < x < 0$. It is clear that $\mathcal{M}$ is itself an infinite-dimensional space. One verifies easily that $P$ satisfies all conditions on a projection. Note that $\mathcal{M}^\perp$ consists of all functions which vanish for $0 < x < \pi$;
every function \( w \) in \( \mathcal{A} \) can be unambiguously decomposed as the sum of a function in \( \mathcal{M} \) (the function \( Pw \)) and a function in \( \mathcal{M}^\perp = (w - Pw) \). Clearly, \( Pw \) and \( w - Pw \) are orthogonal, that is,

\[
\int_{-\pi}^{\pi} (Pw)(w - Pw)dx = 0.
\]

We extend \( u(x) \) to the interval \(-\pi < x < \pi\) and require that \( u \equiv 0 \) in \(-\pi < x < 0\). Thus \( u = Pu \). Then (8.100) becomes

\[
\int_{-\pi}^{\pi} k(x - \zeta)u(\zeta)d\zeta + u(x) = f(x) + g(x), \quad -\pi < x < \pi,
\]

where \( Pu = u, Pf = f, \) and \( Pg = 0 \). If we apply \( P \) to both sides of the new integral equation, we obtain

\[
P \int_{-\pi}^{\pi} k(x - \zeta)u(\zeta)d\zeta + Pu(x) = Pf, \quad -\pi < x < \pi.
\]

Therefore (8.100) has become the equation

\[ PAu = Pf, \]

where the solution is required to satisfy \( Pu = u \). The method of this section is therefore applicable and we may use (8.96) to find an approximation to \( u \) and (8.94) or (8.97) to obtain an upper bound to \( \alpha = \langle f, u \rangle \).

### 8.10 METHOD OF LEAST SQUARES

Consider again the problem (8.11)

\[ Au = f, \quad u \in D_A, \]

where \( A \) is an arbitrary operator. We shall assume that this equation has one and only one solution. We look for an approximate solution of the form

\[ u_n^* = \sum_{i=1}^{n} c_i v_i, \]

where \( v_1, \ldots, v_n \) are \( n \) fixed independent functions. The function \( u_n^* \) would be an exact solution if \( Au_n^* = f \); of course, in general we cannot choose \( c_1, \ldots, c_n \) so that \( Au_n^* = f \), but we can pick these coefficients so that \( \| Au_n^* - f \| \) or \( \| Au_n^* - f \|^2 \) is as small as possible. The method of least squares consists of selecting \( c_1, \ldots, c_n \) so that

\[
\left\| \sum_{i=1}^{n} c_i Av_i - f \right\|^2
\]

is minimum. Setting

\[ g_i = Av_i, \]
we must minimize

\[ \left\| \sum_{i=1}^{n} c_i g_i - f \right\|^2, \]

which is equivalent to finding the projection of \( f \) on the subspace spanned by \( g_1, \ldots, g_n \). According to (8.9), we have the following set of algebraic equations for \( c_1, \ldots, c_n \):

\[ \langle f, g_j \rangle = \sum_{k=1}^{n} c_k \langle g_k, g_j \rangle, \quad j = 1, \ldots, n, \]

or

\[ \langle f, Av_j \rangle = \sum_{k=1}^{n} c_k \langle Av_k, Av_j \rangle, \quad j = 1, \ldots, n. \quad (8.102) \]

If the functions \( Av_1, \ldots, Av_n \) are independent, this set of equations has one and only one solution for \( c_1, \ldots, c_n \). If the functions \( Av_1, \ldots, Av_n \) are dependent, the equations (8.102) are still compatible and there is more than one solution. In any event, the equations (8.102) should be compared with the Ritz-Rayleigh equations (8.23).

By choosing \( \langle Av_i, Av_j \rangle = \delta_{ij} \), the equations (8.102) simplify to

\[ \langle f, Av_i \rangle = c_i, \]

which implies

\[ u_n^* = \sum_{i=1}^{n} \langle f, Av_i \rangle v_i. \]

By assumption \( f \) is in the range of the operator \( A \). If we choose an infinite set \( v_1, \ldots, v_n, \ldots \), so that \( Av_i \) is complete in the range of \( A \), then \( f = \lim_{n \to \infty} f_n \), where \( f_n = \sum_{i=1}^{n} c_i Av_i \) and the \( \{ c_i \} \) are given by (8.102). We then have

\[ u_n^* = \sum_{i=1}^{n} c_i v_i = A^{-1} f_n. \]

If the operator \( A^{-1} \) is continuous, then, since \( f_n \) converges to \( f \); \( u_n^* \) must converge to \( A^{-1} f \), that is, to \( u \). Convergence of the approximate solution \( u_n^* \) to \( u \) can also be proved under other assumptions, but we do not pursue the matter further.

It should be pointed out that the Rayleigh-Ritz method as applied to \( Au = f \) is equivalent to the method of least squares as applied to a certain related equation (if \( A \) is symmetric and positive). Indeed, suppose \( A \) is symmetric and positive; then there exists an unambiguous positive symmetric operator \( A^{1/2} \) (the square root of \( A \)) with the property \( A^{1/2} A^{1/2} = A \). Then \( Au = f \) is equivalent to

\[ A^{1/2} u = A^{-1/2} f, \]
and if we apply the method of least squares to this new equation, we have, from (8.102),

$$\langle A^{-1/2}f, A^{1/2}v_j \rangle = \sum_{k=1}^{n} c_k \langle A^{1/2}v_k, A^{1/2}v_j \rangle, \quad j = 1, \ldots, n,$$

or

$$\langle f, v_j \rangle = \sum_{k=1}^{n} c_k \langle Av_k, v_j \rangle, \quad j = 1, \ldots, n,$$

which are just the Ritz-Rayleigh equations (8.23) for $Au = f$.

**EXERCISES**

8.1 A symmetric, positive operator $A$ is said to be strongly positive if there exists a positive constant $c$ such that

$$\langle Au, u \rangle \geq c\|u\|^2, \quad (8.103)$$

for all $u$ in the domain of definition of $A$. Show that the Ritz-Rayleigh solution (8.29) converges in the mean if $A$ is strongly positive.

8.2 Let $A$ be a symmetric, nonnegative operator on $D_A$ and suppose that $Au = f$ has a solution (which need not be unique since $Au = 0$ may have nontrivial solution). Show that every solution of $Au = f$ maximizes the functional

$$F(v) = \langle f, v \rangle + \langle v, f \rangle - \langle Av, v \rangle.$$

8.3 *The Neumann Problem.* Consider the operator

$$A = -\nabla^2$$

defined on the domain $D_A$ of functions $u(x)$ with continuous second derivatives in the closed region $R$ and satisfying the condition $\partial u/\partial n = 0$ on the boundary $\sigma$ of $R$. Show that $A$ is nonnegative (but not positive) on $D_A$. It is easily seen that the boundary value problem

$$-\nabla^2 u = f, \quad x \in R; \quad \frac{\partial u}{\partial n} = 0, \quad x \text{ on } \sigma, \quad (8.104)$$

has a solution if and only if the consistency condition

$$\int_{R} f \, dx = 0 \quad (8.105)$$

is satisfied. If (8.105) is satisfied, then $u$ is determined only to an additive constant.

We wish to estimate the numerical quantity

$$\alpha = \int_{R} fu \, dx = \int_{R} |\text{grad } u|^2 \, dx$$
associated with (8.104), where we have assumed that (8.105) holds. Clearly \( \alpha \) is independent of the particular solution \( u \) used for (8.104). We rewrite (8.104) as

\[
-\text{div} \: U = f, \quad x \text{ in } R; \quad U \cdot n = 0, \quad x \text{ on } \sigma; \quad U = \text{grad} \: u.
\]

Here \( U \) is a vector function of \( x \). Introduce the inner product

\[
[U, V] = \int_R U \cdot V \: dx \tag{8.106}
\]

in the space of vector functions on \( R \). By using the inequality (8.37) show that

\[
\alpha = \min_{\substack{V_n|\sigma = 0 \\text{div} V = f}} \int_R |V|^2 \: dx. \tag{8.107}
\]

Using the inequality (8.35) with the inner product (8.38), show that

\[
\alpha = \max_v \left( 2 \int_R v f \: dx - \int_R |\text{grad} \: v|^2 \: dx \right), \tag{8.108}
\]

where the admissible scalar functions \( v \) are not required to satisfy the boundary condition \( \partial v / \partial n = 0 \).

### 8.4 The Neumann problem (continued)

Consider the boundary value problem

\[
\nabla^2 u = 0, \quad x \text{ in } R; \quad \frac{\partial u}{\partial n} = f, \quad x \text{ on } \sigma. \tag{8.109}
\]

The necessary and sufficient condition for existence of a solution is

\[
\int_\sigma f \: dS = 0,
\]

which we shall assume to hold. The solution is then determined up to an additive constant. By using methods similar to those in Exercise 8.3 obtain extremal principles for \( \alpha = \int_\sigma f u \: dS = \int_R |\text{grad} \: u|^2 \: dx \).

### 8.5

Consider the operator \( A = -\nabla^2 \) defined for functions \( u(x) \) having continuous second derivatives in a closed region \( R \) and satisfying the condition

\[
\frac{\partial u}{\partial n} + k(x)u = 0 \tag{8.110}
\]

on the boundary \( \sigma \) of \( R \). Here \( k(x) \) is a given, real, positive function defined on \( \sigma \). Show that the operator is positive. Consider the boundary value problem

\[
-\nabla^2 u = f, \quad x \text{ in } R; \quad \frac{\partial u}{\partial n} + k(x)u = 0, \quad x \text{ on } \sigma. \tag{8.111}
\]

Show that the solution is equally well characterized as the function
which yields the maximum of
\[ F(v) = 2 \int_R f v \, dx - \int_R |\nabla v|^2 \, dx - \int_\sigma k v^2 \, dS, \]
among all functions \( v \) with a piecewise continuous first derivative. Note that \( v \) is not required to satisfy the boundary condition, so that (8.111) is the natural boundary condition associated with \( F(v) \).

8.6 Consider (8.111) for two different \( k(x) \), say, \( k_1(x) \) and \( k_2(x) \), where \( k_1(x) \leq k_2(x) \) for \( x \) on \( \sigma \). For the same \( f \) denote the corresponding solutions by \( u_1 \) and \( u_2 \), respectively. Let \( \alpha_1 = \langle f, u_1 \rangle \), \( \alpha_2 = \langle f, u_2 \rangle \). Show that \( \alpha_2 \leq \alpha_1 \). Let \( u_3 \) be a solution of (8.104) and \( u_4 \) of the Dirichlet problem (8.39), with \( \alpha_3 \) and \( \alpha_4 \) defined as \( \langle f, u_3 \rangle \) and \( \langle f, u_4 \rangle \), respectively. Compare \( \alpha_1 \), \( \alpha_2 \), \( \alpha_3 \), and \( \alpha_4 \).

8.7 Consider the biharmonic operator \( \nabla^4 \) defined for functions \( u(x) \) with continuous derivatives of the fourth order in the closed region \( R \) and satisfying the conditions \( u = 0 \) and \( \partial u / \partial n = 0 \) on the boundary \( \sigma \) of \( R \). Show that the operator is positive. Consider the boundary value problem
\[ \nabla^4 u = f, \quad x \in R; \quad u = \frac{\partial u}{\partial n} = 0, \quad x \text{ on } \sigma. \]

Let
\[ \alpha = \int_R f u \, dx = \int_R u \nabla^4 u \, dx = \int_R (\nabla^4 u)^2 \, dx. \]

By introducing the inner product
\[ [v, w] = \int_R (\nabla^2 v)(\nabla^2 w) \, dx, \]
and applying the inequalities (8.35) and (8.37), show that
\[ \alpha = \max_{v|_x = 0, \frac{\partial v}{\partial n}|_x = 0} \left( 2 \int_R f v \, dx - \int_R (\nabla^2 v)^2 \, dx \right), \]
\[ \alpha = \min_{\nabla^2 v = f} \int_R (\nabla^2 v)^2 \, dx. \]

In each case the function \( v = u \) provides the extremal. Find the natural boundary conditions associated with the functional
\[ F(v) = 2 \int_R f v \, dx - \int_R (\nabla^2 v)^2 \, dx. \]

8.8 Consider the two boundary value problems
\[ -\nabla^2 u_1 + k_1(x)u_1 = f(x), \quad x \in R; \quad u_1|_\sigma = 0, \]
\[ -\nabla^2 u_2 + k_2(x)u_2 = f(x), \quad x \in R; \quad u_2|_\sigma = 0, \]
where \( 0 < k_1(x) \leq k_2(x) \). Compare the quantities \( \int_R fu_1 \, dx \) and \( \int_R fu_2 \, dx \).
8.9 The usual method of deriving the Ritz-Rayleigh equations (8.23) is different from the one in the text. Let \( v_1, \ldots, v_n \) be an independent set of functions in \( D_A \); we then try to choose the constants \( c_1, \ldots, c_n \) so as to maximize

\[
F\left( \sum_{i=1}^{n} c_i v_i \right).
\]

Let \( c_1^0, \ldots, c_n^0 \) be the values which yield the maximum and let \( c_i = c_i^0 + \epsilon \eta_i \), where the \( \{\eta_i\} \) are arbitrary complex numbers and \( \epsilon \) is real. Show that by setting

\[
\left( \frac{\partial F}{\partial \epsilon} \right)_{\epsilon=0} = 0,
\]

the equations (8.23) are obtained.

8.10 An integrodifferential equation. Let \( A \) be the operator defined by

\[
Au = -\frac{d^2u}{dx^2} + \int_0^1 xtu(t)dt.
\]

The domain \( D_A \) of \( A \) is the set of all functions \( u(x) \) on \( 0 \leq x \leq 1 \) with a continuous second derivative and satisfying the boundary conditions

\[
u(0) = 0, \quad u'(1) = 0. \quad \text{(8.112)}
\]

(a) Show that \( A \) on \( D_A \) is a symmetric, positive operator. Since \( A \) is real, we can, in what follows, restrict the domain of \( A \) to real functions.

(b) Consider the integrodifferential equation

\[
-u'' + \int_0^1 xtu(t)dt = f(x), \quad 0 < x < 1; \quad u(0) = 0, \quad u'(1) = 0. \quad \text{(8.113)}
\]

Since the integral term is proportional to \( x \), we can easily solve this equation. Show that the one and only solution of (8.113) is

\[
u(x) = \int_0^1 k(x, t)f(t)dt,
\]

where

\[
k(x, t) = g(x \mid t) - \frac{5}{204} (3x - x^3)(3t - t^3).
\]

Here

\[
g(x \mid t) = \begin{cases} x, & 0 < x < t, \\ t, & t < x < 1, \end{cases}
\]

is the Green's function of \(-d^2/dx^2\) subject to the boundary conditions (8.112).
(e) We wish to characterize the solution of (8.113) by an extremal principle and we also want to estimate

$$\alpha = \int_0^1 fu \, dx = \int_0^1 (u')^2 \, dx + \left( \int_0^1 xu \, dx \right)^2.\) 

From (8.16) and part (a) of the present exercise, we have

$$\alpha = \max_{v \in D_A} \left[ 2 \int_0^1 fv \, dx - \int_0^1 (v')^2 \, dx - \left( \int_0^1 xv \, dx \right)^2 \right]. \quad (8.114)$$

and the maximum occurs for \(v = u\). Also, using (8.17),

$$\alpha = \max_{v \in D_A} \frac{\left( \int_0^1 fv \, dx \right)^2}{\int_0^1 (v')^2 \, dx + \left( \int_0^1 xv \, dx \right)^2}, \quad (8.115)$$

the maximum occurring for \(v = cu\).

For the special case \(f = 1\), show that the exact value of \(\alpha\) as calculated from part (b) is 0.295. Use the trial function \(v = x(2 - x)\) in (8.115) and compare with the exact value of \(\alpha\).

(d) Let \(F(v)\) be the functional which appears in (8.114) and let \(h(x)\) be a real function with a piecewise continuous first derivative and satisfying only the boundary condition \(h(0) = 0\). Show that

$$F(u + h) < F(u),$$

for every \(h(x)\) not identically zero. Thus the boundary condition \(u'(1) = 0\) is natural for the functional \(F\). Hence we can extend the class of admissible functions in (8.114) and (8.115) to include functions satisfying only the essential boundary condition at \(x = 0\). Using the trial function \(v = x\), estimate \(\alpha\) from (8.115). Of course, the result is less accurate than the one obtained by using the trial function \(x(2 - x)\), which also satisfies \(v'(1) = 0\).

8.11 Ritz-Rayleigh equations for a general operator. Consider the problem (8.84) for an arbitrary operator. We wish to estimate \(\alpha = \langle u, g \rangle\), where \(g\) is a given element. Let \(w_1, \ldots, w_n\) be \(n\) independent functions in \(D_A\), which span the \(n\)-dimensional subspace \(E_n\); let \(z_1, \ldots, z_n\) be \(n\) independent functions in \(D_A^*\), which span the subspace \(E_n^*\). We look for an approximate solution of (8.84) of the form

$$w = \sum_{i=1}^n c_i w_i,$$

and for an approximate solution of (8.85) in the form

$$z = \sum_{i=1}^n d_i z_i.$$
The coefficients can be calculated by substituting these expressions for \(w\) and \(z\) in (8.87) and requiring that \(F\) be stationary with respect to arbitrary changes in the coefficients. D. S. Jones has shown that this procedure is equivalent to (1) substituting the trial function \(w\) in (8.84) and requiring that both sides have the same projection on \(E_n^*\), and (2) substituting the trial function \(z\) in (8.85) and requiring that both sides have the same projection on \(E_n\). Following this prescription, we find

\[
\sum_{i=1}^{n} c_i \langle Aw_i, z_j \rangle = \langle f, z_j \rangle, \quad j = 1, \ldots, n \tag{8.116}
\]

\[
\sum_{i=1}^{n} d_i \langle w_j, A^* z_i \rangle = \langle w_j, g \rangle, \quad j = 1, \ldots, n. \tag{8.117}
\]

The first set of equations determines \(c_1, \ldots, c_n\) and the second, \(d_1, \ldots, d_n\). The corresponding approximations to \(u\) and \(v\) are given, respectively, by

\[
\tilde{w} = \sum_{i=1}^{n} c_i w_i, \quad \tilde{z} = \sum_{i=1}^{n} d_i z_i.
\]

To estimate \(\alpha\) one should, in theory, substitute \(\tilde{w}\) and \(\tilde{z}\) in (8.87), but actually the three terms \(\langle \tilde{w}, g \rangle, \langle f, \tilde{z} \rangle\), and \(\langle A \tilde{w}, \tilde{z} \rangle\) are all equal. Any pair of approximations \((\tilde{w}, \tilde{z})\) for which these three terms are equal is said to satisfy the extended reciprocity principle. We now show that \((\tilde{w}, \tilde{z})\) as determined from (8.116) and (8.117) satisfies the extended reciprocity principle. Multiply (8.116) by \(d_j\) and sum on the index \(j\) to obtain

\[
\sum_{j=1}^{n} \sum_{i=1}^{n} c_i d_j \langle Aw_i, z_j \rangle = \langle A \tilde{w}, \tilde{z} \rangle = \langle f, \sum_{j=1}^{n} d_j z_j \rangle = \langle f, \tilde{z} \rangle.
\]

Multiply (8.117) by \(c_j\) and sum over \(j\) to find

\[
\sum_{j=1}^{n} \sum_{i=1}^{n} c_j d_i \langle w_j, A^* z_i \rangle = \langle \tilde{w}, A^* \tilde{z} \rangle = \langle \tilde{w}, g \rangle.
\]

Since \(\tilde{w}\) is in \(D_A\) and \(\tilde{z}\) in \(D_{A^*}\), \(\langle A \tilde{w}, \tilde{z} \rangle = \langle \tilde{w}, A^* \tilde{z} \rangle\) and

\[
\langle \tilde{w}, g \rangle = \langle f, \tilde{z} \rangle = \langle A \tilde{w}, \tilde{z} \rangle,
\]

which was to be proved. Thus to estimate \(\alpha\), it suffices to calculate either the \(\{c_i\}\) or \(\{d_i\}\). Suppose, say, we have found \(c_1, \ldots, c_n\) from (8.116); we then set \(\tilde{w} = \sum_{i=1}^{n} c_i w_i\) and from (8.87) and the extended reciprocity principle,

\[
\tilde{\alpha} = \langle \tilde{w}, g \rangle.
\]

Of course, we have no way of predicting whether \(\tilde{\alpha}\) is an upper or lower bound to \(\alpha\).
8.11 EXTREMAL PRINCIPLES FOR EIGENVALUE PROBLEMS ON EUCLIDEAN $n$ SPACE

Let $A$ be an arbitrary symmetric operator on an $n$-dimensional Euclidean space $E_n$ and consider the eigenvalue problem

$$Au = \lambda u.$$  

(8.118)

As was shown in Volume I, p. 160, there are $n$ real, not necessarily distinct, eigenvalues

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n,$$

and corresponding orthonormal eigenvectors $u_1, \ldots, u_n$.

If $u$ is an eigenvector (which need not be normalized) corresponding to the eigenvalue $\lambda$, we can express $\lambda$ in terms of $u$ from (8.118),

$$\lambda = \frac{\langle Au, u \rangle}{\|u\|^2}.$$  

(8.119)

We claim that $\lambda_1$ and $u_1$ can be characterized by a minimum principle and $\lambda_n$ and $u_n$ by a maximum principle. For simplicity we shall assume that $\lambda_1$ and $\lambda_n$ are not degenerate. For any vector $v \neq 0$, we define the Rayleigh quotient as

$$R(v) = \frac{\langle Av, v \rangle}{\|v\|^2}, \quad v \neq 0.$$  

(8.120)

In what follows the restriction $v \neq 0$ is always imposed.

**Theorem 1**

$$\lambda_1 = \min_{v \in E_n} R(v), \quad \text{minimum occurs for } v = cu_1;$$  

(8.121)

$$\lambda_n = \max_{v \in E_n} R(v), \quad \text{maximum occurs for } v = cu_n.$$  

(8.122)

**Proof.** Since the normalized eigenvectors $u_1, \ldots, u_n$ form an orthonormal basis in $E_n$, we can expand any vector $v$ as

$$v = \sum_{i=1}^{n} c_i u_i, \quad c_i = \langle v, u_i \rangle,$$

and, therefore,

$$Av = \sum_{i=1}^{n} \lambda_i c_i u_i.$$

It follows that

$$\langle Av, v \rangle = \sum_{i=1}^{n} \lambda_i |c_i|^2, \quad \|v\|^2 = \sum_{i=1}^{n} |c_i|^2.$$
and, for \( \nu \neq 0 \),
\[
R(\nu) = \frac{\sum_{i=1}^{n} \lambda_i |c_i|^2}{\sum_{i=1}^{n} |c_i|^2}.
\]

Since \( \lambda_1 \leq \lambda_i \) and \( \lambda_i \leq \lambda_n \), we have
\[
\lambda_1 \leq R(\nu) \leq \lambda_n.
\]

Moreover, \( R(cu_k) = \lambda_1 \) and \( R(cu_n) = \lambda_n \), which completes the proof.

**Remarks.** 1. Instead of finding the extremum of the quotient \( R(\nu) \), we can equally well find the extremum of its numerator \( \langle Av, \nu \rangle \) subject to the constraint \( \|\nu\| = 1 \).

2. If \( \lambda_1 \) is degenerate, say, \( \lambda_1 = \lambda_2 \cdots = \lambda_k \), the minimum of \( R(\nu) \) is still \( \lambda_1 \) but now the minimum occurs for any linear combination of eigenfunctions corresponding to that value of \( \lambda \), that is, for \( \nu = c_1 u_1 + \cdots + c_k u_k \).

Next we wish to characterize intermediate eigenvalues by extremal principles involving the Rayleigh quotient, the admissible vectors being constrained in a suitable manner. We introduce the following notation: \( M_k \) is the linear manifold spanned by the first \( k \) eigenvectors of \( A \); that is, \( M_k \) is the set of all linear combinations of \( u_1, \ldots, u_k \). Then \( M_k^\perp \), the orthogonal complement of \( M_k \), is the linear manifold spanned by \( u_{k+1}, \ldots, u_n \).

**Theorem 2**
\[
\begin{align*}
\lambda_k &= \min_{v \in M_{k-1}^\perp} R(\nu), \quad \text{minimum occurs for } v = cu_k; \quad (8.123) \\
\lambda_k &= \max_{v \in M_k} R(\nu), \quad \text{maximum occurs for } v = cu_k. \quad (8.124)
\end{align*}
\]

**Proof.** Again we write for any \( \nu \),
\[
v = \sum_{i=1}^{n} c_i u_i, \quad \text{where } c_i = \langle v, u_i \rangle.
\]

If \( v \in M_{k-1}^\perp \), then \( c_1 = \cdots = c_{k-1} = 0 \) and \( v = \sum_{i=k}^{n} c_i u_i \), so that
\[
R(\nu) = \frac{\sum_{i=k}^{n} \lambda_i |c_i|^2}{\sum_{i=k}^{n} |c_i|^2} \geq \lambda_k;
\]

moreover, \( cu_k \) is in \( M_{k-1}^\perp \) and \( R(cu_k) = \lambda_k \). This proves (8.123). If \( v \in M_k \),
then \( c_{k+1} = \cdots = c_n = 0 \) and \( v = \sum_{i=1}^{k} c_i u_i \); therefore,

\[
R(v) = \frac{\sum_{i=1}^{k} \lambda_i |c_i|^2}{\sum_{i=1}^{k} |c_i|^2} \leq \lambda_k.
\]

The vector \( cu_k \) is in \( M_k \) and \( R(cu_k) = \lambda_k \), so that (8.124) has been proved.

In practice, these characterizations of \( \lambda_k \) are somewhat unsatisfactory, since they require a knowledge of either \( M_{k-1}^\perp \) or \( M_k \), that is, of all the lower eigenvectors or all the higher ones. We want to have a characterization of \( \lambda_k \) that is independent of any explicit knowledge of other eigenvectors. Two such characterizations are possible: one as a minimax, the other as a maximin. We prove one of these and leave the other to the reader. Variants of the theorem appear in the literature, but we have chosen forms for these theorems which can be readily extended to infinite-dimensional spaces.

**Theorem 3 (Maximin Theorem).** Let \( E_{k-1} \) be any subspace of dimension \( k - 1 \), and let \( \alpha \) be the minimum of the Rayleigh quotient subject to the constraint \( v \in E_{k-1}^\perp \). Then \( \alpha \) depends on the choice of \( E_{k-1} \) and will be denoted by \( \alpha(E_{k-1}) \). The theorem states

\[
\lambda_k = \max_{\text{all choices of } E_{k-1}} \alpha(E_{k-1}),
\]

or

\[
\lambda_k = \max_{\text{all choices } v \in E_{k-1}} \min_{\text{of } E_{k-1}} \frac{\langle Av, v \rangle}{\|v\|^2}. \tag{8.125}
\]

**Theorem 4 (Minimax Theorem).** Let \( E_k \) be any subspace of dimension \( k \), and let \( \beta \) be the maximum of the Rayleigh quotient with \( v \in E_k \). Then \( \beta \) depends, of course, on the choice of \( E_k \) and will be denoted by \( \beta(E_k) \). We have

\[
\lambda_k = \min_{\text{all choices of } E_k} \beta(E_k),
\]

or

\[
\lambda_k = \min_{\text{all choices } v \in E_k} \max_{\text{of } E_k} \frac{\langle Av, v \rangle}{\|v\|^2}. \tag{8.126}
\]

**Proof.** We will prove Theorem 3. First we observe that from (8.123) \( \alpha(M_{k-1}) = \lambda_k \). Therefore, the maximum of \( \alpha \) over all choices of \( E_{k-1} \) is larger than or equal to \( \lambda_k \). To prove the opposite inequality, we exhibit for each choice of \( E_{k-1} \) a vector \( v \) orthogonal to \( E_{k-1} \) with the property \( R(v) \leq \lambda_k \). Since the space \( M_k \) is of higher dimension than \( E_{k-1} \), there must exist a non-zero vector \( v \) in \( M_k \) which is orthogonal to \( E_{k-1} \). By (8.124) we have \( R(v) \leq \lambda_k \). 

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Therefore, for each $E_{k-1}$ there is a vector $v$ in $E_{k-1}^1$ such that $R(v) \leq \lambda_k$. Hence the minimum of $R(v)$, that is, $\alpha(E_{k-1})$, is surely smaller than or equal to $\lambda_k$. Hence the maximum of $\alpha(E_{k-1})$ over all choices of $E_{k-1}$ is smaller than or equal to $\lambda_k$. Together with the opposite inequality proved first, this completes the proof of Theorem 3.

8.12 EIGENVALUE PROBLEMS IN HILBERT SPACE

In trying to extend the extremal principles of the preceding section to symmetric operators defined either on the whole of a Hilbert space $\mathcal{A}$ or, in the case of an unbounded operator, on a linear manifold $D_{\mathcal{A}}$ dense in $\mathcal{A}$, we again expect the Rayleigh quotient (8.120) to play a central role. Since the spectral properties of operators on infinite-dimensional spaces are so much more complicated than on finite-dimensional spaces, some limitations will have to be imposed on the operators under consideration. One obvious difficulty is that the eigenvalues may now extend all the way to $+\infty$ or $-\infty$ if the operator is unbounded. For instance, the operator $A = -d^2/dx^2$ on the domain $D_{\mathcal{A}}$ of functions $u(x)$ with two continuous derivatives in $0 \leq x \leq 1$ and satisfying $u(0) = u(1) = 0$ gives rise to eigenvalues $\lambda_n = n^2\pi^2$, and clearly $\lambda_n \to +\infty$ as $n \to \infty$. We can therefore only hope to have extremal principles which work their way up from the lower end of the spectrum (that is, starting from $\lambda_1$). We shall consider only semibounded operators, that is, operators which are either bounded below (as in the example just given) or bounded above. (for instance, the operator $-A$, where $A$ is the operator of the example just given).

**DEFINITION.** A symmetric operator $A$ on $D_{\mathcal{A}}$ is said to be **bounded below** if there exists a constant $k$ (possibly negative) such that

$$\langle Av, v \rangle \geq k\|v\|^2, \quad \text{for all } v \in D_{\mathcal{A}}. \quad (8.127)$$

A symmetric operator $A$ on $D_{\mathcal{A}}$ is **bounded above** if there exists a constant $k$ such that

$$\langle Av, v \rangle \leq k\|v\|^2, \quad \text{for all } v \in D_{\mathcal{A}}. \quad (8.128)$$

**REMARK.** In Chapter 2 we defined a bounded operator $A$ as follows:

There exists a constant $c$ such that $\|Av\| \leq c\|v\|$ for all $v$ in $D_{\mathcal{A}}$. By Schwarz's inequality we have

$$|\langle Av, v \rangle| \leq \|Av\| \|v\| \leq c\|v\|^2,$$

$$-c\|v\|^2 \leq \langle Av, v \rangle \leq c\|v\|^2,$$

so that a bounded operator is bounded above and below.

In the remainder of this chapter we consider only semibounded operators. **All theorems will be proved only for operators bounded below**, since, mutatis mutandis.
mutandis, we can easily state similar theorems for operators bounded above.

With $A$ bounded below, the Rayleigh quotient $R(v) = \langle Av, v \rangle / \|v\|^2$, $v \neq 0$, $v \in D_A$ is also bounded below. Therefore as $v$ runs through $D_A$, $v \neq 0$, there exists a greatest lower bound, say $m_A$, to $R(v)$. Then we have

$$R(v) \geq m_A, \quad v \in D_A, \quad v \neq 0,$$

and, to each $\varepsilon > 0$, there exists $v \neq 0$ in $D_A$ such that

$$R(v) \leq m_A + \varepsilon.$$

This suggests that $m_A$ might be the lowest eigenvalue of $A$; unfortunately this need not be true. Unlike the finite-dimensional case, the greatest lower bound of $R(v)$ may not be assumed for any $v$ in $D_A$; that is, there may not exist a function $v \in D_A$ such that $R(v) = m_A$. This difficulty is associated with the existence of a continuous spectrum and is illustrated by the following simple example.

Let $A$ be $L^2(0, 1)$ and let $A$ be defined as

$$Au = f(x)u(x), \quad u(x) \in A,$$

where $f(x)$ is a fixed, continuous, positive, increasing function on $0 \leq x \leq 1$. To be specific, take $f(x) = 1 + x$. We easily see that the operator $A$ is bounded yet has no eigenvalues. The Rayleigh quotient is

$$R(v) = \frac{\int_0^1 (1 + x)|v|^2 \, dx}{\int_0^1 |v|^2 \, dx}.$$

Since

$$\int_0^1 (1 + x)|v|^2 \, dx \geq \int_0^1 |v|^2 \, dx,$$

we have

$$R(v) \geq 1, \quad \text{for all } v \neq 0.$$

Consider the function

$$v_\varepsilon(x) = \begin{cases} 1, & 0 < x < \varepsilon, \\ 0, & \varepsilon < x < 1; \end{cases}$$

then

$$R(v_\varepsilon) = 1 + \frac{\varepsilon}{2},$$

and we can make $R(v_\varepsilon)$ as close to 1 as we please. In conjunction with the inequality $R(v) \geq 1$, this means that the greatest lower bound of $R(v)$ is $m_A = 1$. Yet there is no function $v$ for which $R(v) = 1$. Thus there is no
function which minimizes $R(v)$. In our particular problem we have a continuous spectrum, $1 \leq \lambda \leq 2$, which corresponds to the interval $f_{\text{min}} \leq \lambda \leq f_{\text{max}}$. To obtain an extremal principle we have to postulate the existence of a function $v$ which yields a minimum to $R(v)$; this then eliminates the possibility of a continuous part in the lower end of the spectrum.

**Theorem 1.** Let $A$ on $D_A$ be a symmetric operator bounded below and let $m_A$ be the greatest lower bound of $R(v)$. If there exists a function $u$ in $D_A$ for which $R(u) = m_A$, then $u$ is an eigenfunction corresponding to the lowest eigenvalue of $A$ and this lowest eigenvalue is $m_A$.

**Proof.** By assumption $R(v)$ is a minimum for $v = u$ and therefore for any $\eta \in D_A$ and any real number $\varepsilon$, we have

$$R(u + \varepsilon \eta) \geq R(u) = m_A.$$

It follows that

$$\left[ \frac{d}{d\varepsilon} R(u + \varepsilon \eta) \right]_{\varepsilon=0} = 0.$$

Performing the required calculation, we find

$$\langle \eta, Au \rangle + \langle Au, \eta \rangle - m_A [\langle \eta, u \rangle + \langle u, \eta \rangle] = 0, \quad \eta \in D_A.$$

Since $i\eta$ also belongs to $D_A$, we may substitute $i\eta$ for $\eta$ to obtain

$$\langle \eta, Au \rangle - \langle Au, \eta \rangle - m_A [\langle \eta, u \rangle - \langle u, \eta \rangle] = 0,$$

or, adding to the preceding equation,

$$\langle \eta, Au \rangle - m_A \langle \eta, u \rangle = 0,$$

that is,

$$\langle \eta, Au - m_A u \rangle = 0.$$

This equation holds for all $\eta$ in $D_A$, and $D_A$ is dense in $\mathcal{A}$; therefore,

$$Au - m_A u = 0$$

and $u$ is an eigenfunction corresponding to the eigenvalue $m_A$. It remains only to prove that $A$ has no eigenvalue lower than $m_A$. Let $\lambda$ be an eigenvalue with eigenfunction $\varphi$. Then $A\varphi = \lambda \varphi$ and $\lambda = [\langle A\varphi, \varphi \rangle/\|\varphi\|^2] = R(\varphi)$. Since the minimum of $R$ is $m_A$, we have $\lambda \geq m_A$, which completes the proof.

**Theorem 2.** Let $A$ on $D_A$ be a symmetric operator bounded above and let $M_A$ be the least upper bound of $R(v)$. If there exists a function $u$ in $D_A$ for which $R(u) = M_A$, then $u$ is an eigenfunction corresponding to the largest eigenvalue of $A$ and this largest eigenvalue is $M_A$.

**Proof.** The operator $-A$ is bounded below. By applying Theorem 1, we find that $-M_A$ is the lowest eigenvalue of $-A$ and hence $M_A$ is the largest eigenvalue of $A$.  

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In applications it is relatively easy to determine if $A$ is bounded above or below or both. The difficulty that arises is in trying to establish whether or not the least upper bound (or greatest lower bound) of the Rayleigh quotient is actually attained for some function $u$ in $D_A$. For two particularly important classes of operators, the question raised in the preceding sentence is easily answered. We consider first the class of symmetric, completely continuous operators. We recall that $A$ is completely continuous if it maps bounded sets into compact sets (see Section 2.10). Typically such an operator is an integral operator generated by a Hilbert-Schmidt kernel or by a kernel with a sufficiently weak singularity (see, for instance, Section 6.6).

Suppose then that $A$ is symmetric and completely continuous. As was seen in Chapter 3, $A$ gives rise in general to both positive and negative eigenvalues. We denote negative eigenvalues by $\mu_i^- (\mu_i^- \leq \mu_{i+1}^-)$ and their normalized eigenfunctions by $\varphi_i^-$; positive eigenvalues by $\mu_i^+ (\mu_i^+ \geq \mu_{i+1}^+)$ and their normalized eigenfunctions by $\varphi_i^+$. Zero may or may not be an eigenvalue; if it is not an eigenvalue it is a limit point of eigenvalues. Let $N$ be the null space of $A$, that is, $N$ is the linear manifold of all eigenfunctions corresponding to the zero eigenvalue; we let $P$ be the projection operator on $N$. The principal result of Chapter 3 is that $A$ has a complete set of eigenfunctions. Thus, if $v$ is any element in the Hilbert space,

$$v = \sum_i \langle v, \varphi_i^- \rangle \varphi_i^- + Pv + \sum_i \langle v, \varphi_i^+ \rangle \varphi_i^+,$$

$$Av = \sum_i \mu_i^- \langle v, \varphi_i^- \rangle \varphi_i^- + \sum_i \mu_i^+ \langle v, \varphi_i^+ \rangle \varphi_i^+.$$

Therefore,

$$R(v) = \frac{\sum \mu_i^- |\langle v, \varphi_i^- \rangle|^2 + \sum \mu_i^+ |\langle v, \varphi_i^+ \rangle|^2}{\|v\|^2}.$$

Clearly,

$$\mu_i^- \leq R(v) \leq \mu_i^+,$$  \hspace{1cm} (8.129)

and since $R(\varphi_i^-) = \mu_i^-$, $R(\varphi_i^+) = \mu_i^+$, both bounds in (8.129) are actually attained.

If $A$ is a positive completely continuous operator, then all eigenvalues are positive and 0 is a limit point of eigenvalues. Then

$$0 < R(v) \leq \mu_i^+$$

and the maximum of $R(v)$ is attained for $v = \varphi_i^+$ but the greatest lower bound of $R(v)$ is not attained for any nonzero $v$.

We now turn to the second class of operators of special importance. Let $A$ be a differential operator defined on a domain $D_A$ characterized by certain homogeneous boundary conditions. We assume that $A$ is bounded below and
self-adjoint} and that it has a completely continuous inverse (generated by a Green’s function). Then $A$ has a discrete set of eigenvalues.

$$\lambda_1 \leq \lambda_2 \leq \lambda_3 \cdots,$$

with

$$\lim_{n \to \infty} \lambda_n = \infty,$$

and its eigenfunctions $u_1(x), \ldots, u_n(x), \ldots$ form a complete set. If $v \in D_A$, we have

$$Av = \sum_{i=1}^{\infty} c_i u_i$$

$$\langle Av, u_j \rangle = c_j = \langle v, Au_j \rangle = \lambda_j \langle v, u_j \rangle.$$ 

Thus

$$R(v) = \frac{\sum_{i=1}^{\infty} \lambda_i |\langle v, u_i \rangle|^2}{\sum_{i=1}^{\infty} |\langle v, u_i \rangle|^2}$$

Clearly,

$$R(v) \geq \lambda_1,$$

and since $R(u_1) = \lambda_1$, we have

$$\min_{v \in D_A} R(v) = \lambda_1.$$

One can develop extremal principles like Theorems 2, 3, and 4 of Section 8.11 for both classes of operators just described. In what follows, we restrict ourselves to the second class of operators. The proofs are similar to those of Section 8.11 and are therefore omitted.

**Theorem 3.** Let $M_k$ be the linear manifold spanned by $u_1, \ldots, u_k$. Then

$$\lambda_k = \min_{v \in M_k} R(v), \quad \text{minimum occurs for } v = cu_k;$$

$$\lambda_k = \max_{v \in M_k} R(v), \quad \text{maximum occurs for } v = cu_k.$$

We also have characterizations of $\lambda_k$ which do not refer explicitly to lower eigenfunctions.

**Theorem 4.** Let $E_k$ be a $k$-dimensional subspace of $D_A$. Then

$$\lambda_k = \min_{\text{all choices } v \in E_k} \max_{\text{of } E_k} R(v),$$

$$\lambda_k = \max_{\text{all choices } v \in E_k} \min_{\text{of } E_k} R(v).$$
Theorem 4 enables us to compare the eigenvalues of certain operators, but first we need a definition. We write $A \preceq B$ if both $A$ and $B$ are symmetric and

$$D_A \supset D_B \quad \text{and} \quad \langle Av, v \rangle \leq \langle Bv, v \rangle, \quad \text{for all } v \in D_B.$$

As an immediate conclusion of this definition and of the first part of Theorem 4, we have the following theorem.

**Theorem 5 (Comparison Theorem).** Let $A \preceq B$ and let each operator have a complete set of eigenfunctions. Moreover, let their eigenvalues (with proper regard to multiplicity) be

$$\lambda_1^B \leq \lambda_2^B \leq \cdots \leq \lambda_n^B \leq \cdots,$$

$$\lambda_1^A \leq \lambda_2^A \leq \cdots \leq \lambda_n^A \leq \cdots,$$

respectively, with $\lim_{n \to \infty} \lambda_n^A = \infty$, $\lim_{n \to \infty} \lambda_n^B = \infty$. Then

$$\lambda_i^A \leq \lambda_i^B, \quad \text{for each } i.$$

**Ritz-Rayleigh Method**

In the Ritz-Rayleigh method, we estimate the lowest $n$ eigenvalues $\lambda_1, \ldots, \lambda_n$ of $A$ on $D_A$ by explicitly calculating the $n$ eigenvalues $v_1, \ldots, v_n$ of the part of $A$ in $E_n$, an $n$-dimensional subspace of $D_A$. Let $P$ be the projection operator on $E_n$; by definition, the part of $A$ in $E_n$ is the operator $PA$ restricted to $E_n$. The eigenvalues $\{v_i\}$ are therefore defined from

$$PAu = vu, \quad u \in E_n. \quad (8.130)$$

**Theorem 6**

$$\lambda_k \leq v_k, \quad k = 1, \ldots, n. \quad (8.131)$$

**Proof.** Let $F_k$ be a $k$-dimensional subspace of $D_A$, $k \leq n$. Then, by Theorem 4,

$$\lambda_k = \min_{F_k \subset E_n} \max_{u \in F_k} \frac{\langle Au, u \rangle}{\|u\|^2} \leq \min_{F_k \subset E_n} \max_{u \in F_k} \frac{\langle Au, u \rangle}{\|u\|^2} = \min_{F_k \subset E_n} \max_{u \in F_k} \frac{\langle Pu, u \rangle}{\|u\|^2} = \min_{F_k \subset E_n} \max_{u \in F_k} \frac{\langle PAu, u \rangle}{\|u\|^2} = v_k.$$

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To determine \( \{v_k\} \) explicitly, we observe that the vector equation (8.130) is equivalent to
\[
(PAu, v_j) = v(u, v_j), \quad j = 1, \ldots, n,
\]
(8.132)
where \( v_1, \ldots, v_n \) is a basis for \( E_n \).

Then since the solution of (8.130) is of the form \( u = \sum_{i=1}^{n} c_i v_i \), we have
\[
\sum_{i=1}^{n} c_i \langle Av_i, v_j \rangle = v \sum_{i=1}^{n} c_i \langle v_i, v_j \rangle, \quad j = 1, \ldots, n.
\]
(8.133)
These are \( n \) homogeneous linear equations for \( c_1, \ldots, c_n \). Nontrivial solutions are possible only if the determinant of the matrix \( \langle Av_j, v_j \rangle - v \langle v_i, v_j \rangle \) vanishes. This will yield \( n \) roots \( v_1 \leq \ldots \leq v_n \), each of which is an upper bound to the corresponding \( \lambda \).

**EXAMPLE**

We consider the following one-dimensional example in some detail:
\[
-\frac{d^2u}{dx^2} = \lambda u, \quad 0 < x < 1; \quad u(0) = 0, \quad u(1) + u'(1) = 0, \quad (8.134)
\]
which can be written in the form
\[
Au = \lambda u, \quad u \in D_A,
\]
where \( A \) is the operator \(-d^2/dx^2\) on the domain \( D_A \) of all functions \( u(x) \) which have a continuous second derivative and such that \( u(0) = 0 \) and \( u(1) + u'(1) = 0 \). Clearly, \( D_A \) is a linear manifold dense in \( \mathcal{L}_2(0, 1) \). The operator \( A \) on \( D_A \) is symmetric since
\[
\langle Au, v \rangle = -\int_0^1 u'' \tilde{v} \, dx = \int_0^1 u \tilde{v}'' \, dx = \langle u, Av \rangle.
\]
Further,
\[
\langle Au, u \rangle = -\int_0^1 u'' \bar{u} \, dx = \int_0^1 |u'|^2 \, dx + |u^2(1)|,
\]
so that \( A \) is positive on \( D_A \) and its eigenvalues are necessarily positive.

The system (8.134) is entirely equivalent to the integral equation obtained by introducing the Green’s function \( g(x | \xi) \) satisfying
\[
-\frac{d^2g}{dx^2} = \delta(x - \xi), \quad 0 < x, \xi < 1; \quad g \bigg|_{x=0} = 0, \quad \left(g + \frac{dg}{dx}\right)_{x=1} = 0.
\]
An easy calculation gives
\[
g(x | \xi) = \frac{1}{2} x \xi (2 - x \xi),
\]
where \( x_- = \min(x, \xi) \), \( x_+ = \max(x, \xi) \). Therefore, (8.134) is equivalent to
\[
u(x) = \lambda \int_0^1 g(x \mid \xi) u(\xi) d\xi,
\]
or
\[
Gu = \mu u, \quad \mu = \frac{1}{\lambda}, \quad G = \text{integral operator generated by } g. \quad (8.135)
\]

Now \( G \) is the inverse of a positive operator and hence is positive itself. Therefore, it gives rise to a set of positive eigenvalues
\[
\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \geq \cdots; \quad \lim_{n \to \infty} \mu_n = 0,
\]
and the corresponding eigenfunctions form a complete set.

It follows that the eigenvalues of (8.134) form an increasing sequence
\[
\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots; \quad \lim_{n \to \infty} \lambda_n = \infty.
\]

If we want to estimate \( \lambda_1 = 1/\mu_1 \) we have two variational principles both of which give upper bounds to \( \lambda_1 \). Applying Theorem 1 to (8.134) we find
\[
\lambda_1 = \min_{v \in D_A} \frac{\langle v, Av \rangle}{\|v\|^2} = \min_{v \in D_A} \frac{-\int_0^1 v\ddot{v} \ dx}{\int_0^1 |v|^2 \ dx},
\]
or
\[
\lambda_1 = \min_{v \in D_A} \frac{\int_0^1 |v'|^2 \ dx + |v|^2(1)}{\int_0^1 |v|^2 \ dx}. \quad (8.136)
\]

For the operator \( G \) we have
\[
\mu_1 = \frac{1}{\lambda_1} = \max_{\|u\|^2} \frac{\langle Gu, u \rangle}{\|u\|^2} = \max_{\|u\|^2} \frac{\int_0^1 \int_0^1 g(x \mid \xi) u(x) \bar{u}(\xi) dx \ d\xi}{\int_0^1 |u|^2 \ dx}.
\]

Therefore,
\[
\lambda_1 = \min \frac{\int_0^1 |u|^2 \ dx}{\int_0^1 \int_0^1 g(x \mid \xi) u(x) \bar{u}(\xi) dx \ d\xi}. \quad (8.137)
\]

As anticipated both (8.136) and (8.137) yield upper bounds to \( \lambda_1 \).
One can use the Ritz-Rayleigh method or variation-iteration (see p. 231, Volume I) in connection with these variational principles. Let us look at variation-iteration; let \( f_0 \) be an arbitrary function and define successively

\[
f_1 = Gf_0, \ldots, f_k = Gf_{k-1}, \ldots.
\]

From the definition of \( G \) we have also

\[
Af_k = f_{k-1}, \quad f_k \in D_A, \quad k = 1, 2, \ldots.
\]

The Schwarz constants \( a_k \) are defined from

\[
a_k = \langle f_{k-i}, f_i \rangle,
\]

a definition which is easily seen to be independent of the index \( i \). With

\[
\theta_{k+1} = \frac{a_{k+1}}{a_k},
\]

we have shown (p. 233, Volume I),

\[
\theta_k \uparrow \mu_1, \quad \text{if } \langle f_0, u_1 \rangle \neq 0,
\]

where \( u_1 \) is the eigenfunction of (8.134) corresponding to \( \lambda_1 \). We have

\[
\theta_{2k+1} = \frac{\langle f_{k+1}, f_k \rangle}{\langle f_k, f_k \rangle} = \frac{\langle Gf_k, f_k \rangle}{\langle f_k, f_k \rangle}, \tag{8.138}
\]

\[
\theta_{2k} = \frac{\langle f_k, f_k \rangle}{\langle f_k, f_{k-1} \rangle} = \frac{\langle f_k, f_k \rangle}{\langle f_k, Af_k \rangle}. \tag{8.139}
\]

Thus \( \theta_{2k} \) is the reciprocal of the Rayleigh quotient of \( f_k \) corresponding to the differential equation formulation, whereas \( \theta_{2k+1} \) is the Rayleigh quotient of \( f_k \) corresponding to the integral equation formulation. Since \( \theta_k \) is monotonically increasing, \( \theta_{2k+1} \) provides a better lower bound to \( \mu_1 \) than does \( \theta_{2k} \); this is not surprising, since \( \theta_{2k+1} \) involves calculating \( Gf_k \), which amounts to an additional iteration. We then have

\[
\frac{1}{\theta_1} \geq \frac{1}{\theta_2} \geq \cdots \geq \frac{1}{\theta_{2k}} \geq \frac{1}{\theta_{2k+1}} \geq \cdots \geq \frac{1}{\mu_1} = \lambda_1.
\]

To see the effectiveness of the procedure let us start with the very crude trial function \( f_0 = 1 \). Then the first iterate \( f_1 \) is defined from

\[
-f_1'' = 1, \quad f_1(0) = 0, \quad f_1'(1) + f_1(1) = 0,
\]

so that

\[
f_1(x) = \frac{1}{2}x^2 - \frac{1}{2}x^2.
\]

The second iterate \( f_2 \) is defined from

\[
-f_2'' = f_1, \quad f_2(0) = 0, \quad f_2'(1) + f_2(1) = 0,
\]
which yields

\[ f_2 = \frac{x^4}{24} - \frac{x^3}{8} + \frac{7}{48} x. \]

A simple calculation based on (8.138) and (8.139) gives

\[ \theta_1 = \frac{5}{24}, \quad \theta_2 = \frac{6}{25}, \quad \theta_3 = \frac{163}{672}. \]

Thus \(1/\theta_3\) gives the following upper bound to \(\lambda_1\):

\[ \lambda_1 \leq \frac{672}{163} = 4.123. \]

To find a lower bound to \(\lambda_1\), we recall the bilinear formula (3.65), which states

\[ \sum_{i=1}^{\infty} \left( \frac{1}{\lambda_i} \right)^n = \int_0^1 k_n(x, x) dx, \quad n = 1, 2, \ldots, \quad (8.140) \]

where

\[ k_n(x, \xi) = g(x | \xi) \]

and \(k_n(x, \xi)\) is the \(n\)th iterated kernel of \(g\). In our case,

\[ k_1(x, x) = \frac{1}{2}x(2 - x), \quad \int_0^1 k_1(x, x) dx = 1/3, \]

\[ k_2(x, x) = \int_0^1 k(x, t) k(t, x) dt = \int_0^1 k^2(x, t) dt = \frac{x^4}{6} - \frac{2}{3} x^3 + \frac{7}{12} x^2, \quad \int_0^1 k_2(x, x) dx = 11/180. \]

From (8.140), we then obtain, since all \(\{\lambda_i\}\) are positive,

\[ \lambda_1 \geq 3 \quad \text{and} \quad \lambda_1^2 \geq 180/11. \]

This last inequality combined with the previous upper bound for \(\lambda_1\) gives the enclosure

\[ 4.045 \leq \lambda_1 \leq 4.123. \]

From (8.134) we find that \(\lambda_1\) is the smallest positive root of \(\tan \sqrt{\lambda} = -\sqrt{\lambda}\). By consulting a table of trigonometric functions, one finds the exact value

\[ \lambda_1 = 4.08. \]

### 8.13 Lower Bounds to Eigenvalues

Let \(A\) on \(D_A\) be a self-adjoint operator bounded below. We assume that either \(A^{-1}\) or \((A - \theta I)^{-1}\), for some real \(\theta\), is completely continuous. Then
the eigenvalues of $A$ form an increasing sequence

$$
\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots; \quad \lim_{n \to \infty} \lambda_n = \infty.
$$

Upper bounds to the $\lambda$'s can be found by the Rayleigh-Ritz procedure [see (8.131)]. It is more difficult to establish computational methods for calculating lower bounds. It is tempting to try to use the maximin characterization of Theorem 4 of Section 8.12, but this requires estimating the Rayleigh quotient on the infinite-dimensional subspace $E^1_{k-1}$, which is no easy matter. Instead we describe a method, the method of intermediate problems, originally due to Weinstein and later modified, first by Aronszajn and then by Bazley and Fox. The method begins with the decomposition of the operator $A$ as the sum of two operators $A_0$ and $C$, where $A_0$ is of the same type as $A$ and $C$ is a positive operator. Thus we assume that $A$ may be written

$$
A = A_0 + C, \quad (8.141)
$$

where $A_0$ has known eigenvalues $\{\lambda^0_k\}$ and eigenfunctions $\{\varphi^0_k\}$. The operator $A_0$ is called the base operator. The domain of $A$ is the intersection of the domains of $A_0$ and $C$.

The decomposition (8.141) with $C > 0$ guarantees that $A_0 \leq A$ and therefore by the comparison theorem (Theorem 5, Section 8.12)

$$
\lambda^0_k \leq \lambda_k,
$$

which immediately provides us with some crude lower bounds to $\lambda_k$. Our goal is to systematically improve these lower bounds by solving intermediate problems; from the terminology used it is clear that we intend to consider intermediate problems for which the eigenvalues can be calculated accurately (preferably, exactly!).

We shall construct a sequence of operators $C_n$ with the property $C_n \geq 0$, $C \geq C_{n+1} \geq C_n$, such that the eigenvalues of $A_0 + C_n$ increase monotonically to those of $A = A_0 + C$. Let $p_1, \ldots, p_n, \ldots$ be a complete set of elements in $D_C$, the domain of $C$. Let $P_n$ be the projection on the subspace spanned by the first $n$ elements $p_1, \ldots, p_n$. We then define

$$
C_n = C^{1/2} P_n C^{1/2}, \quad (8.142)
$$

where $C^{1/2}$ is the unambiguously defined positive square root of the operator $C$. It will turn out that we will not need to calculate $C^{1/2}$; all we have to know is that $C^{1/2}$ exists, and this is guaranteed by a theorem in Hilbert space which we do not prove here.

We observe that $C_n$ is symmetric and nonnegative on $D_C$; indeed

$$
\langle C_n u, v \rangle = \langle C^{1/2} P_n C^{1/2} u, v \rangle,
$$
and, since $C^{1/2}$ is symmetric,

$$\langle C_n u, v \rangle = \langle P_n C^{1/2} u, C^{1/2} v \rangle = \langle C^{1/2} u, P_n C^{1/2} v \rangle$$

$$= \langle u, C^{1/2} P_n C^{1/2} v \rangle = \langle u, C_n v \rangle,$$

$$\langle C_n u, u \rangle = \langle P_n C^{1/2} u, C^{1/2} u \rangle \geq 0.$$  

Also,

$$\langle C_{n+1} u, u \rangle = \langle P_{n+1} C^{1/2} u, C^{1/2} u \rangle \geq \langle P_n C^{1/2} u, C^{1/2} u \rangle = \langle C_n u, u \rangle.$$  

Moreover, it is clear that $\langle C_n u, u \rangle \leq \langle Cu, u \rangle$. Therefore, $C_{n+1} \geq C_n$ and $A \geq A_0 + C_{n+1} \geq A_0 + C_n$. The eigenvalues of the intermediate problem

$$(A_0 + C_n)u = \lambda u,$$ (8.143)

therefore furnish lower bounds to those of $A$ (by the comparison theorem). By (8.10), we have

$$P_n C^{1/2} u = \sum_{k=1}^{n} \left\langle C^{1/2} u, \sum_{j=1}^{n} a_{kj} p_j \right\rangle p_k,$$

where $\{a_{kj}\}$ is the inverse matrix of $\{\langle p_k, p_j \rangle\}$. Thus

$$C^{1/2} P_n C^{1/2} u = \sum_{k=1}^{n} \left\langle u, \sum_{j=1}^{n} a_{kj} C^{1/2} p_j \right\rangle C^{1/2} p_k.$$  

Now let

$$h_j = C^{1/2} p_j,$$ (8.144)

which implies

$$C^{1/2} P_n C^{1/2} u = \sum_{k=1}^{n} \left\langle u, \sum_{j=1}^{n} a_{kj} h_j \right\rangle h_k,$$ (8.145)

where $\{a_{kj}\}$ is the inverse matrix of $\{\langle C^{-1/2} h_k, C^{-1/2} h_j \rangle\}$. This latter matrix has elements which can be rewritten as $\langle h_k, C^{-1} h_j \rangle$. The intermediate problem (8.143) therefore becomes

$$A_0 u + \sum_{k=1}^{n} \sum_{j=1}^{n} a_{kj} \langle u, h_j \rangle h_k = \lambda u.$$ (8.146)

The elements $\{h_k\}$ are at our disposal; we now choose them so that (8.146) can be reduced to an algebraic problem (which we can then presumably solve to any required degree of precision). We let

$$h_k = \varphi_k^0,$$  

where $\varphi_k^0$ is the $k$th eigenfunction of $A_0$. This special choice of $h_k$ reduces (8.146) to

$$A_0 u + \sum_{k=1}^{n} \sum_{j=1}^{n} a_{kj} \langle u, \varphi_j^0 \rangle \varphi_k^0 = \lambda u,$$ (8.147)
where we recall that \( \{a_{kj}\} \) is the inverse matrix of \( \{\langle \varphi^0_k, C^{-1} \varphi^0_j \rangle \} \). Since \( \{a_{kj}\} \) is a symmetric matrix and \( A_0 \) is symmetric, the operator which appears on the left side of (8.147) is necessarily symmetric. It is clear that \( \varphi_{n+1}^0, \varphi_{n+2}^0, \ldots \) are all eigenfunctions of (8.147) corresponding respectively to \( \lambda_{n+1}^0, \lambda_{n+2}^0, \ldots \). The remaining eigenfunctions of (8.147) must therefore be orthogonal to \( \varphi_{n+1}^0, \varphi_{n+2}^0, \ldots \) and are therefore linear combinations of \( \varphi_1^0, \ldots, \varphi_n^0 \). To find these additional eigenfunctions, we set

\[
u = \sum_{m=1}^{n} c_m \varphi_m^0,
\]

and substitute in (8.147). This yields

\[
\sum_{m=1}^{n} c_m \lambda_m^0 \varphi_m^0 + \sum_{k=1}^{n} \sum_{j=1}^{n} a_{kj} c_j \varphi_k^0 = \lambda \sum_{m=1}^{n} c_m \varphi_m^0,
\]
or

\[
c_m \lambda_m^0 + \sum_{j=1}^{n} a_{mj} c_j = \lambda c_m, \quad m = 1, \ldots, n. \tag{8.149}
\]

Equation (8.149) is an algebraic eigenvalue problem which yields \( n \) eigenvalues \( \lambda_1', \ldots, \lambda_n' \), the \( n \) roots of the determinantal equation

\[
\begin{vmatrix}
\lambda_1^0 + a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\
a_{21} & \lambda_2^0 + a_{22} - \lambda & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & \lambda_n^0 + a_{nn} - \lambda
\end{vmatrix} = 0. \tag{8.150}
\]

To each of the roots \( \lambda_i' \) corresponds an eigenvector \( (c_1', \ldots, c_n') \) from (8.149). Substituting in (8.148) we have \( n \) additional eigenfunctions for (8.147).

Our principal interest is in the eigenvalues of (8.147). These consist of the sequence

\[
\lambda_{n+1}^0 \leq \lambda_{n+2}^0 \leq \cdots
\]

and of the \( n \) real roots \( \lambda_1', \ldots, \lambda_n' \) of (8.150). We arrange all these in a single monotonically increasing sequence (note that some of the \( \{\lambda_i'\} \) may exceed \( \lambda_{n+1}^0 \)).

We then have all the eigenvalues of the intermediate problem (8.143) or (8.147) in increasing order. We denote this sequence by

\[
\lambda_1^{(n)} \leq \lambda_2^{(n)} \leq \cdots,
\]

and, by the comparison theorem,

\[
\lambda_k^{(n)} \leq \lambda_k.
\]
Some useful information is obtained even by considering the simple case
\( n = 1 \). Then (8.150) reduces to
\[
\lambda'_1 = \lambda^0_1 + a_{11} = \lambda^0_1 + \frac{1}{\langle \phi^0_1, C^{-1} \phi^0_1 \rangle}.
\] (8.151)
Thus the eigenvalues of \( A_0 + C_1 \) are
\[
\lambda'_1, \lambda'_2, \lambda'_3, \ldots,
\]
and if (as must be determined in each individual case)
\[
\lambda'_1 < \lambda^0_2,
\]
then
\[
\lambda'_1 \leq \lambda_1.
\] (8.152)
This gives a lower bound to \( \lambda_1 \), which is better than \( \lambda'_1 \).

As an example, consider the problem
\[
-\frac{d^2}{dx^2} u + \rho(x) u = \lambda u, \quad 0 < x < 1, \quad u(0) = u(1) = 0,
\] (8.153)
where \( \rho(x) \) is a given positive continuous function on \( 0 < x < 1 \). The
base operator \( A_0 \) is \( -d^2/dx^2 \) on the domain \( D_0 \) of functions satisfying
\( u(0) = u(1) = 0 \). Its eigenvalues and eigenfunctions are easily calculated:
\[
\lambda^0_k = k^2 \pi^2, \quad k = 1, 2, \ldots
\]
\[
\phi^0_k = \sqrt{2} \sin k\pi x.
\]
The operator \( C \) is a multiplicative operator:
\[
C u = \rho(x) u,
\]
and is defined for all \( u \) in \( L^2(0, 1) \); it is clear that \( C \) is a positive operator.
We have
\[
a_{11} = \frac{1}{\langle \phi^0_1, C^{-1} \phi^0_1 \rangle} = \left[ 2 \int_0^1 \frac{\sin^2 \pi x}{\rho(x)} \, dx \right]^{-1}.
\]
Consider the special case \( \rho(x) = \sin \pi x \). The eigenvalues of (8.147) are
\( \lambda_1 \leq \lambda_2 \cdots \). According to (8.151) and (8.152) we have
\[
\lambda_1 \geq \lambda^0_1 + a_{11} = \pi^2 + \frac{\pi}{4},
\]
if \( \lambda^0_1 + a_{11} < \lambda^0_2 \), which is certainly the case. An upper bound to \( \lambda_1 \) can be
found by using the trial function \( v = \sin \pi x \) in the Ritz-Rayleigh quotient.
This gives
\[
\lambda_1 \leq \pi^2 + 2 \int_0^1 \sin^3 \pi x \, dx = \pi^2 + \frac{8}{3\pi}.
\]
In conjunction with the previous inequality we have the enclosure interval for \( \lambda_1 \),

\[ \pi^2 + 0.79 \leq \lambda_1 \leq \pi^2 + 0.85. \]

**Exercises**

8.12 Let \( k(x, \xi) \) be a real, symmetric Hilbert-Schmidt kernel on \( a \leq x, \xi \leq b \), which generates a positive operator \( K \) with eigenvalues \( \{\mu_i\} \). Then from (3.66) we have

\[ \mu_1 \leq \left[ \int_a^b k_n(x, x)dx \right]^{1/n}, \]

where \( k_n \) is the \( n \)th iterated kernel of \( k \). Show that

\[ \lim_{n \to \infty} \left[ \int_a^b k_n(x, x)dx \right]^{1/n} = \mu_1. \]

8.13 The eigenvalue problem for the normal modes of a beam with built-in ends is

\[ \frac{d^4u}{dx^4} = \lambda u, \quad 0 < x < 1; \quad u(0) = u'(0) = u(1) = u'(1) = 0. \quad (8.154) \]

(a) Without solving the differential equation, show that the eigenvalues are real and strictly positive; show that the eigenfunctions can be chosen as real functions.

(b) Find the Green’s function \( g(x | \xi) \) for \( D^4 \) with the boundary conditions above. [Hint: Use a causal fundamental solution and adjust appropriately (the causal solution is \( H(x - \xi)(x - \xi)^3/6 \).]

(c) Translate (8.154) into an integral equation with \( \mu = 1/\lambda \). Show that \( \mu = 0 \) is not an eigenvalue of the integral equation.

(d) Using variation-iteration with \( f_0 = 1 \), calculate the Schwarz quotients

\[ \frac{\langle f_1, f_0 \rangle}{\langle f_0, f_0 \rangle}, \frac{\langle f_1, f_1 \rangle}{\langle f_0, f_0 \rangle}, \frac{\langle f_2, f_1 \rangle}{\langle f_1, f_1 \rangle}. \]

Thus obtain upper bounds for \( \lambda_1 \).

(e) Obtain a lower bound for \( \lambda_1 \) by using

\[ \lambda_1 \geq \frac{1}{\int_0^1 g(x \mid x)dx}, \quad \lambda_1 \geq \left[ \int_0^1 g_2(x \mid x)dx \right]^{1/2}, \]

where \( g_2(x \mid \xi) \) is the iterated kernel of \( g \).

8.14 Let

\[ Au = -[xu'(x)]' \]

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and consider the eigenvalue problem

\[ Au = \lambda x u, \quad 0 < x < 1; \quad u(1) = 0, \quad \lim_{x \to 0^+} xu'(x) = 0. \quad (8.155) \]

The domain \( D_A \) of \( A \) consists of real functions \( u(x) \) with a continuous second derivative in \( 0 < x < 1 \), and satisfying the above boundary conditions. The Green's function for \(-(d/dx)[x(d/dx)] \) subject to the above boundary conditions is

\[
g(x \mid \xi) = \begin{cases} 
-\log \xi, & 0 < x < \xi; \\
-\log x, & \xi < x < 1.
\end{cases} \quad (8.156)
\]

Therefore (8.155) is equivalent to the integral equation

\[
u(x) = \lambda \int_0^1 x g(x \mid \xi) u(\xi) d\xi,
\]

which, in turn, can be reduced to

\[
\mu v(x) = \int_0^1 k(x, \xi) z(\xi) d\xi,
\]

where \( \mu = 1/\lambda, \ k(x, \xi) = \sqrt{x \xi} g(x \mid \xi), \ z(x) = \sqrt{x} u(x). \) The eigenvalues of (8.155) are positive with \( \lambda_n \to \infty. \) In fact, \( \lambda_n = \beta_n^2 \), where \( \beta_n \) is the \( n \)th positive root of \( J_0(x) \). From tables we find

\[
\lambda_1 = 5.78.
\]

(a) Show that

\[
\lambda_1 = \min_{\nu \in D_A} \frac{\int_0^1 x (\nu')^2 \, dx}{\int_0^1 x \nu^2 \, dx}, \quad (8.157)
\]

\[
\lambda_1 = \min \frac{\int_0^1 x v^2 \, dx}{\int_0^1 \int_0^1 k(x, \xi) \nu(x) \nu(\xi) \, dx \, d\xi}.
\]

Use the trial function \( v = 1 - x^2 \) in (8.157) to obtain

\[
\lambda_1 \leq 6.
\]

(b) Consider the iteration scheme

\[
A f_{k+1} = x f_k, \quad 0 < x < 1, \quad f_{k+1} \in D_A.
\]

Starting with an arbitrary \( f_0 \), we can successively determine \( f_1, f_2, \ldots \), by solving an inhomogeneous differential equation with homogeneous boundary conditions. In terms of the Green's function (8.156), we have

\[
f_{k+1}(x) = \int_0^1 g(x \mid \xi) \xi f_k(\xi) d\xi.
\]
Setting

\[ h_k(x) = \sqrt{xf_k(x)}, \]

we have

\[ h_{k+1}(x) = \int_0^1 k(x, \xi)h_k(\xi)d\xi, \]

which is the equivalent iteration scheme for the symmetric integral operator \( K \) whose kernel is \( k(x, \xi) \). Thus the Schwarz constants (8.138) and (8.139), defined from the iteration using \( K \), are

\[
\theta_{2k+1} = \frac{\langle h_{k+1}, h_k \rangle}{\langle h_k, h_k \rangle} = \frac{\int_0^1 xf_{k+1}f_k \, dx}{\int_0^1 xf_k^2 \, dx},
\]

\[
\theta_{2k} = \frac{\langle h_k, h_k \rangle}{\langle h_k, h_{k-1} \rangle} = \frac{\int_0^1 xf_k^2 \, dx}{\int_0^1 xf_kf_{k-1} \, dx}.
\]

We have

\[
\frac{1}{\theta_1} \geq \cdots \geq \frac{1}{\theta_{2k}} \geq \frac{1}{\theta_{2k+1}} \geq \cdots \geq \frac{1}{\mu_1} = \lambda_i.
\]

Starting with \( f_0(x) = 1 \), find \( f_1, f_2, \theta_1, \theta_2, \) and \( \theta_3 \). The last of these gives

\[ \lambda_1 \leq 180/31 = 5.81, \]

which improves the result of part (a).

(c) Obtain lower bounds for \( \lambda_1 \) from

\[ \lambda_1 \geq \frac{1}{\int_0^1 k(x, x)dx}, \]

and

\[ \lambda_1 \geq \left[ \int_0^1 k_2(x, x)dx \right]^{1/2}. \]

This latter bound turns out to be

\[ \lambda_1 \geq 5.66, \]

which in conjunction with the result of part (b) determines \( \lambda_1 \) fairly accurately.
8.15 Consider the eigenvalue problem \( Au = \lambda u \) for the integrodifferential operator of Exercise 8.10; that is,

\[
-u''(x) + \int_0^1 xtu(t)dt = \lambda u(x), \quad 0 < x < 1; \quad u(0) = u'(1) = 0.
\]  

(8.158)

Since \( A \) on \( D_A \) is symmetric and positive, all eigenvalues are real and positive.

(a) By observing that the integral term in (8.158) is proportional to \( x \), show that the eigenvalues \( \lambda = \alpha^2 \) are the positive roots of the equation

\[
\tan \alpha = \alpha + \frac{\alpha^3}{3} - \alpha^5.
\]

Sketch the functions \( \tan \alpha \) and \( \alpha + (\alpha^3/3) - \alpha^5 \) and locate graphically the intersections. From tables obtain an approximate value of \( \lambda_1 = \alpha_1^2 \), the smallest eigenvalue.

(b) Translate (8.158) into an eigenvalue problem for an integral equation by using the fact [proved in part (b) of Exercise 8.10] that \( A^{-1} \) is an integral operator with kernel

\[
k(x, t) = x - \frac{5}{204} (3x - x^3)(3t - t^3).
\]

It follows that the eigenfunctions of (8.158) form a complete set.

(c) From the theory of Section 8.12, we have

\[
\lambda_1 = \min_{v \in D_A} \frac{\int_0^1 (v')^2 \, dx + \left( \int_0^1 xv \, dx \right)^2}{\int_0^1 v^2 \, dx}.
\]  

(8.159)

Use the trial function \( v = x(2 - x) \), which is in \( D_A \) to obtain the upper bound

\[
\lambda_1 \leq 45/16 = 2.81.
\]

From the inequality

\[
\lambda_1 \geq \frac{1}{\int_0^1 k(x, x)dx},
\]

obtain a crude lower bound to \( \lambda_1 \). Find a better lower bound from

\[
\lambda_1 \geq \left[ \int_0^1 k_2(x, x)dx \right]^{1/2},
\]

where \( k_2 \) is the iterated kernel of \( k \).
(d) Show that the boundary condition \( u'(1) = 0 \) is natural, so that
\[
\lambda_1 = \min_{u(0) = 0} R(v),
\]
where \( R(v) \) is the Rayleigh quotient of (8.159). Use the trial function \( v(x) = x \) [which does not satisfy \( u'(1) = 0 \)] to obtain the upper bound
\[
\lambda_1 \leq 10/3,
\]
which is, of course, cruder than the upper bound found in part (c), but requires simpler calculations.

8.16 Consider the eigenvalue problem
\[
Au = -u''(x) + \int_0^1 x t u(t) dt = \lambda u(x), \quad 0 < x < 1; \quad u(0) = u(1) = 0,
\]
(8.160)
which differs from (8.158) only in the boundary condition at \( x = 1 \).

(a) Show that \( A \) on \( D_A \) is symmetric and positive.

(b) Obtain a transcendentall equation for \( \lambda = \sigma^2 \) and locate the eigenvalues graphically. Estimate the lowest eigenvalue \( \lambda_1 \).

(c) Obtain an upper bound for \( \lambda_1 \) by using the trial function \( v = x(1 - x) \) in the Rayleigh quotient for \( A \).

(d) Translate (8.160) into a pure integral equation with kernel \( k(x, t) \).
You will need the Green’s function for \( -d^2/dx^2 \) with the boundary conditions \( u(0) = u(1) = 0 \).

(e) Obtain lower bounds for \( \lambda_1 \) from
\[
\lambda_1 \geq \frac{1}{\int_0^1 k(x, x) dx},
\]
\[
\lambda_1 \geq \left( \frac{1}{\int_0^1 k_2(x, x) dx} \right)^{1/2}.
\]

8.17 Consider the three eigenvalue problems below for the same bounded region \( R \) with boundary \( \sigma \):

(I) \[-\nabla^2 u = \lambda u, \quad x \text{ in } R; \quad u = 0, \quad x \text{ on } \sigma;\]

(II) \[-\nabla^2 u = \lambda u, \quad x \text{ in } R; \quad \frac{\partial u}{\partial n} = 0, \quad x \text{ on } \sigma;\]

(III) \[-\nabla^2 u = \lambda u, \quad x \text{ in } R; \quad \frac{\partial u}{\partial n} + hu = 0, \quad x \text{ on } \sigma.\]

In the last problem \( h \) is a given nonnegative function of position on \( \sigma \).
Note that \( h = 0 \) yields the second problem, and \( h \to \infty \), in some sense yields the first problem.
We may regard these problems as characterizing the normal frequen-
cies and modes of a membrane stretched over the plane region $R$
subject to the following boundary conditions: (a) edge fixed, for (I);
(b) edge free, for (II); and (c) edge elastically supported, for (III).

Returning now to a region $R$ in $n$ dimensions, we shall denote the
eigenvalues of problem (I) by $\lambda_k^{(1)}$, those of (II) by $\lambda_k^{(2)}$, and those of (III) by $\lambda_k^{(3)}(h)$. The Rayleigh quotient for problems (I) and (II) is

$$H(v) = \frac{\int_R |\text{grad } v|^2 \, dx}{\int_R v^2 \, dx},$$

and for problem III

$$I_h(v) = \frac{\left[\int_R |\text{grad } v|^2 \, dx + \int_{\partial R} \text{hu}^2 \, dS\right]}{\int_R v^2 \, dx}.$$

Show that the boundary condition $\partial u/\partial n = 0$ is natural for $H$ and that
the condition $(\partial u/\partial n) + hu = 0$ is natural for $I$. We can therefore characterize the eigenvalues of (I), (II), and (III) by the following extremal principles (see Theorem 4, Section 8.12),

$$\lambda_k^{(1)} = \max_{v_n \in E_{k-1}} \min_{v \in E_{k-1}} H(v), \quad (8.161)$$

$$\lambda_k^{(2)} = \max_{v_n \in E_{k-1}} \min_{v \in E_{k-1}} H(v), \quad (8.162)$$

$$\lambda_k^{(3)}(h) = \max_{v_n \in E_{k-1}} \min_{v \in E_{k-1}} I_h(v). \quad (8.163)$$

We observe that no boundary condition is imposed on $v$ in the second
and third principles, since the desired boundary condition is natural
for the functional under consideration. Now clearly

$$\min_{v \in E_{k-1}} H(v) \leq \min_{v_n \in E_{k-1}} H(v),$$

since the set of admissible functions on the left includes the one on the
right. Since this is true for each choice of $E_{k-1}$, we have

$$\lambda_k^{(2)} \leq \lambda_k^{(1)}.$$ 

If $h_2 \geq h_1$, then $I_{h_2}(v) \geq I_{h_1}(v)$, and therefore

$$\lambda_k^{(3)}(h_2) \geq \lambda_k^{(3)}(h_1).$$

In particular if $h$ is not identically zero,

$$\lambda_k^{(3)}(h) \geq \lambda_k^{(2)}.$$
Moreover we can compare (8.163) with (8.161). We have

\[
\min_{v \in E^1} I_h(v) \leq \min_{v \in E^2} I_h(v) = \min_{v \in E^2} H(v).
\]

Therefore,

\[
\lambda^{(3)}_k(h) \leq \lambda^{(1)}_k.
\]

Consolidating our results, we find (with \( h_1 \leq h_2 \))

\[
\lambda^{(2)}_k \leq \lambda^{(3)}_k(h_1) \leq \lambda^{(3)}_k(h_2) \leq \lambda^{(1)}_k.
\]  
(8.164)

These results have a simple physical interpretation for a membrane: The normal frequencies of a membrane increase as the edge constraint is stiffened.

8.18 Consider the eigenvalue problems

\[
-u'' + p_1(x)u = \lambda u, \quad a < x < b, \quad u(a) = u(b) = 0
\]

\[
-u'' + p_2(x)u = \lambda u, \quad a < x < b, \quad u(a) = u(b) = 0
\]

and denote their eigenvalues by \( \{\lambda^{(1)}_k\} \) and \( \{\lambda^{(2)}_k\} \), respectively. Show that if \( p_1(x) \geq p_2(x) \), then

\[
\lambda^{(1)}_k \geq \lambda^{(2)}_k.
\]

8.19 The method of intermediate problems can also be used to find upper bounds to eigenvalues. Let \( A \) have the decomposition

\[
A = B_0 - C,
\]

where \( B_0 \) is a base operator of the same type as \( A \) and \( C \) is a positive operator. We then construct, as in (8.142) and (8.143), the intermediate problems

\[
(B_0 - C_n)u = \lambda u,
\]

which now provide upper bounds to the eigenvalues of \( A \).
In terms of spherical coordinates (see Figure A.1), Laplace's equation
\[ \nabla^2 u = 0 \] takes the form
\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial u}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2} = 0.
\]

We shall look for harmonic functions which are of the form
\[ u^*(r, \theta, \phi) = R(r)Y(\theta, \phi). \]

Substituting in the differential equation and labeling the separation constant \( \lambda \), we find that \( Y \) satisfies the eigenvalue problem
\[ SY = \lambda \sin \theta Y, \quad (A.1) \]
where $S$ is the differential operator defined by

$$S = -rac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) - \frac{1}{\sin \theta} \frac{\partial^2}{\partial \phi^2}.$$  \hspace{1cm} (A.2)

The same equation (A.1) arises when separating Helmholtz’s equation in spherical coordinates. From the definition (5.12), one easily verifies that $S$ is formally self-adjoint.

Naturally we expect some boundary conditions to be associated with (A.1). If the equation is to hold on the entire surface of a sphere $(0 < \theta < \pi, 0 < \phi < 2\pi)$, we must require that $Y$ be continuous and have a continuous gradient on this surface. The following boundary conditions ensure these properties:

$$Y(\theta, 0+) = Y(\theta, 2\pi-), \quad \frac{\partial Y}{\partial \phi} (\theta, 0+) = \frac{\partial Y}{\partial \phi} (\theta, 2\pi-),$$

$Y$ finite at $\theta = 0$, $\theta = \pi$.

The first two conditions reflect the fact that the planes $\phi = 0$ and $\phi = 2\pi$ are the same physical plane. The condition of finiteness at $\theta = 0$ and $\theta = \pi$ must be imposed because these values of $\theta$ are singular for the differential equation; it turns out that some solutions are infinite at $\theta = 0$ or $\theta = \pi$ and such solutions must be discarded. By the usual arguments, it can be shown that the eigenvalues of $S$ are real and nonnegative, and that eigenfunctions corresponding to different eigenvalues are orthogonal on the surface of the unit sphere. To calculate the eigenfunctions we look first for those eigenfunctions which are separable functions of $\theta$ and $\phi$,

$$Y(\theta, \phi) = \Theta(\theta)\Phi(\phi).$$

Substituting in (A.1), we find

$$-\frac{\sin \theta (\sin \theta \Theta')'}{\Theta} - \lambda \sin^2 \theta = \frac{\Phi''}{\Phi} = -\mu.$$  \hspace{1cm} (A.3)

The boundary conditions on the $\phi$ equation are

$$\Phi(0) = \Phi(2\pi), \quad \Phi'(0) = \Phi'(2\pi).$$

Thus to obtain nontrivial solutions of the $\phi$ equation we must have

$$\mu = m^2, \quad m = \ldots, -2, -1, 0, 1, 2, \ldots,$$

and the corresponding solution $\Phi_m$ is

$$\Phi_m = e^{im\phi}.$$  \hspace{1cm} (A.4)

Turning to the $\theta$ equation, we have

$$-(\sin \theta \Theta')' - \lambda \sin \theta \Theta + \frac{m^2}{\sin \theta} \Theta = 0, \quad 0 < \theta < \pi.$$  \hspace{1cm} (A.5)
The substitution $x = \cos \theta$ transforms the interval $0 < \theta < \pi$ into $-1 < x < 1$ and the differential equation becomes

$$-\frac{d}{dx} \left[ (1 - x^2) \frac{d\Theta}{dx} \right] + \frac{m^2}{1 - x^2} \Theta = \lambda \Theta, \quad -1 < x < 1. \quad (A.4)$$

With $m$ a given integer, this is an eigenvalue problem in the parameter $\lambda$. For most values of $\lambda$ there is no solution which is finite at $x = 1$ and $x = -1$ (these values of $x$ correspond, respectively, to $\theta = 0$ and $\theta = \pi$). Only if $\lambda = n(n + 1), n = |m|, |m| + 1, \ldots$ can we find a solution which is finite at $x = 1$ and at $x = -1$. This solution is the associated Legendre function

$$P^{|m|}_n(\cos \theta).$$

In particular, if $m = 0$, the permissible values of $n$ are $0, 1, 2, \ldots$ and the corresponding eigenfunctions are the ordinary Legendre polynomials

$$P_n(\cos \theta) = P^0_n(\cos \theta).$$

If we now return to (A.1), the only eigenvalues are

$$\lambda = n(n + 1), \quad n = 0, 1, 2, \ldots,$$

and to the eigenvalue $n(n + 1)$ correspond the $2n + 1$ eigenfunctions

$$Y^m_n(\theta, \varphi) = e^{im\varphi} P^{|m|}_n(\cos \theta), \quad |m| \leq n.$$

For instance, for $\lambda = 2$ we have the three eigenfunctions

$$e^{i\varphi} P^1_1(\cos \theta), \quad e^{-i\varphi} P^1_1(\cos \theta), \quad P^1_1(\cos \theta).$$

As mentioned earlier, it follows from (A.1) and the boundary conditions that the set

$$Y^m_n(\theta, \varphi), \quad |m| \leq n, \quad n = 0, 1, 2, \ldots$$

is an orthogonal set on the surface of the unit sphere. Moreover, the set can be shown to be complete and therefore there can be no other eigenfunctions of (A.1). Recalling that the element of area on the surface is $\sin \theta \, d\theta \, d\varphi$, we have

$$\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin \theta \, Y^m_n(\theta, \varphi) \overline{Y^\beta_x(\theta, \varphi)} = 0 \quad \text{unless } \beta = n \text{ and } x = m.$$

The normalization integral is defined as

$$N_{m,n} = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin \theta \, |Y^m_n(\theta, \varphi)|^2,$$

and it can be shown that

$$N_{m,n} = \frac{4\pi}{2n + 1} \frac{(n + |m|)!}{(n - |m|)!}. \quad (A.5)$$
If $f(\theta, \varphi)$ is any function regular on the surface of the unit sphere, we have the expansion

$$f(\theta, \varphi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} f_{m,n} Y_n^m(\theta, \varphi),$$

where

$$f_{m,n} = \frac{1}{N_{m,n}} \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin \theta f(\theta, \varphi) \overline{Y}_n^m(\theta, \varphi). \tag{A.6}$$

In particular, if $f$ is independent of $\varphi$, we have the expansion in ordinary Legendre polynomials

$$f(\theta) = \sum_{n=0}^{\infty} f_n P_n(\cos \theta),$$

where

$$f_n = \frac{2n + 1}{2} \int_0^{\pi} f(\theta) P_n(\cos \theta) \sin \theta \, d\theta.$$

### Spherical Expansion of the Free-Space Green's Function for Laplace's Equation

Let a unit point source be placed at the point $(r_0, \theta_0, \varphi_0)$, where $r_0 \neq 0$, $\theta_0 \neq 0$, and $\theta_0 \neq \pi$. Then

$$E(x | x_0) = \frac{1}{4\pi |x - x_0|} = E(r, \theta, \varphi | r_0, \theta_0, \varphi_0)$$

satisfies

$$-\frac{\partial}{\partial r} \left( r^2 \frac{\partial E}{\partial r} \right) + \frac{1}{\sin \theta} SE = \frac{\delta(r - r_0)\delta(\theta - \theta_0)\delta(\varphi - \varphi_0)}{\sin \theta}, \tag{A.7}$$

where $S$ is the operator (A.2).

For $r \neq r_0$, we may expand $E$ in a series of spherical harmonics:

$$E = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} E_{m,n} Y_n^m(\theta, \varphi),$$

where $E_{m,n}$ depends on $r, r_0, \theta_0, \varphi_0$, and, from (A.6),

$$E_{m,n} = \frac{1}{N_{m,n}} \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin \theta E \overline{Y}_n^m(\theta, \varphi).$$

To find $E_{m,n}$, we multiply both sides of (A.7) by

$$\frac{1}{N_{m,n}} \sin \theta \overline{Y}_n^m(\theta, \varphi)$$
and integrate from $\theta = 0$ to $\pi$ and from $\varphi = 0$ to $2\pi$. Thus

$$- \frac{d}{dr} \left( r^2 \frac{dE_{m,n}}{dr} \right) + \frac{1}{N_{m,n}} \int_0^{2\pi} d\varphi \int_0^\pi d\theta \, Y_n^m(\theta, \varphi) SE = \frac{1}{N_{m,n}} Y_n^m(\theta_0, \varphi_0).$$

Integrating by parts, using the fact that $S$ is symmetric and that

$$SY_n^m(\theta, \varphi) = n(n+1) \sin \theta \, Y_n^m(\theta, \varphi),$$

we obtain the ordinary differential equation for $E_{m,n}$:

$$- \frac{d}{dr} \left( r^2 \frac{dE_{m,n}}{dr} \right) + n(n+1) E_{m,n} = \frac{1}{N_{m,n}} Y_n^m(\theta_0, \varphi_0) \delta(r - r_0).$$

The functions $r^n$ and $r^{-n-1}$ are linearly independent solutions of the homogeneous equation, the first of which is bounded at $r = 0$ and the other at $r = \infty$. By standard arguments for one-dimensional problems, we find

$$E_{m,n} = \frac{Y_n^m(\theta_0, \varphi_0)}{(2n+1)N_{m,n}} r_< r_>^{-n-1},$$

where $r_<$ = min$(r, r_0)$ and $r_>$ = max$(r, r_0)$. Hence

$$E(x | x_0) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} r_< r_>^{-n-1} \frac{Y_n^m(\theta, \varphi) Y_n^m(\theta_0, \varphi_0)}{(2n+1)N_{m,n}}. \quad \text{(A.8)}$$

If the source is placed at a point where $\theta_0 = 0$, that is, on the positive $z$ axis, then (A.8) simplifies. Indeed we have

$$Y_n^m(0, \varphi_0) = e^{-im\varphi_0} P_n^{|m|}(1),$$

where

$$P_n^{|m|}(1) = \begin{cases} 0, & m \neq 0, \\ 1, & m = 0, \end{cases}$$

and therefore

$$E(x | x_0) = \frac{1}{4\pi} \sum_{n=0}^{\infty} r_< r_>^{-n-1} P_n(\cos \theta) ; \quad \text{source at } \theta_0 = 0, r_0. \quad \text{(A.9)}$$

If we now return the source to its arbitrary position in (A.8), the angle $\theta$ in (A.9) becomes the angle $\gamma$ between the points $(1, \theta_0, \varphi_0)$ and $(1, \theta, \varphi)$. An easy calculation shows that

$$\cos \gamma = \cos \theta \cos \theta_0 + \sin \theta \sin \theta_0 \cos(\varphi - \varphi_0), \quad \text{(A.10)}$$

so that (A.8) can be written alternatively as

$$\frac{1}{4\pi} \sum_{n=0}^{\infty} r_< r_>^{-n-1} P_n(\cos \gamma).$$
Comparison with (A.8) then yields the addition theorem for Legendre functions:

\[ P_n(\cos \gamma) = \sum_{m=-n}^{n} \frac{e^{im(\varphi-\varphi_0)}P_n^{|m|}(\cos \theta)P_n^{|m|}(\cos \theta_0)}{(n + |m|)!/(n - |m|)!}, \tag{A.11} \]

where \( \cos \gamma \) is given by (A.10).

A particular case of (A.9) deserves special mention. If the source is placed at \( r_0 = 1, \theta_0 = 0 \), we have

\[ \frac{1}{[1 + r^2 - 2r \cos \theta]^{1/2}} = \sum_{n=0}^{\infty} r^n P_n(\cos \theta), \quad r < 1, \tag{A.12} \]

\[ \frac{1}{[1 + r^2 - 2r \cos \theta]^{1/2}} = \sum_{n=0}^{\infty} r^{-n-1} P_n(\cos \theta), \quad r > 1. \tag{A.13} \]

By differentiating with respect to \( r \), we find, from (A.12),

\[ \frac{1 - r^2}{[1 + r^2 - 2r \cos \theta]^{3/2}} = \sum_{n=0}^{\infty} (2n + 1)r^n P_n(\cos \theta), \quad r < 1. \tag{A.14} \]
Consider

$$\int_a^b f(x)e^{-th(x)}dx = I(t),$$  \hspace{1cm} (B.1)

where $a$ and $b$ are given real numbers, $h(x)$ a real-valued function, $f(x)$ a complex-valued function, and $t$ a positive number. We shall be interested in obtaining computationally useful formulas for $I(t)$ when $t$ is large. We proceed without regard to rigor. When $t$ is large, the exponential function in the integrand is peaked about the minimum value of $h(x)$ and the principal contribution to $I$ for large $t$ stems from a neighborhood of the point $c$ at which $h(x)$ is smallest. Suppose then that $h(x)$ has a minimum at the interior point $c$, $a < c < b$, and that $h''(c) > 0$. We can then replace the integral from $a$ to $b$ by one from $c - \delta$ to $c + \delta$, where $\delta$ is a small positive number. The value of $I(t)$ is then approximately given, for large $t$, by

$$f(c) \int_{c-\delta}^{c+\delta} e^{-th(c)}e^{-th''(c)(x-c)^2/2} \, dx$$

$$= 2f(c)e^{-th(c)} \int_0^{\delta[th''(c)/2]^{1/2}} e^{-u^2} \left[ \frac{2}{th''(c)} \right]^{1/2} \, du.$$  

$$= 2f(c)e^{-th(c)} \left[ \frac{2}{th''(c)} \right]^{1/2} \int_0^{\infty} e^{-u^2} \, du$$

$$= \left[ \frac{2\pi}{th''(c)} \right]^{1/2} f(c)e^{-th(c)}.$$  

We therefore write

$$\int_a^b f(x)e^{-th(x)} \, dx \sim \left[ \frac{2\pi}{th''(c)} \right]^{1/2} f(c)e^{-th(c)}, \hspace{1cm} (B.2)$$
a result which is independent of the location of the end points as long as 
\( h(x) \) has a single minimum in the interior of the interval. The expression on 
the right represents the leading term in an asymptotic expansion of \( I(t) \) for large \( t \).

Higher-order terms can be found by examining \( f \) and \( h \) more closely. The 
method can be modified to handle cases where \( f(c) \) vanishes or \( h''(c) = 0 \).

If the minimum of \( h(x) \) in \( a \leq x \leq b \) occurs at an end point, say, at \( x = a \), 
the derivative of \( h \) may either vanish or be positive at \( x = a \). If \( h'(a) = 0 \) and 
\( h''(a) > 0 \), we use one half of the value given by (B.2). If \( h'(a) > 0 \), we may 
replace the original interval of integration by one from \( a \) to \( a + \delta \) and set 
\( h(x) = h(a) + h'(a)(x - a) \) to obtain, after an obvious change of variables,

\[
\int_a^b f(x)e^{-th(x)}dx \sim \frac{e^{-th(a)}f(a)}{th'(a)}.
\]  

(B.3)

Compared to (B.2), the exponential decreases more quickly from the point 
where \( h \) has a minimum, so that the integral is smaller for large \( t \). Again one 
can find additional terms in the expansion without great difficulty.

As a special case of (B.1), consider the Laplace integral

\[
L(t) = \int_0^\infty e^{-tx}f(x)dx,
\]  

(B.4)

where \( f(x) \) is \( O(e^{\varepsilon x}) \) at infinity, so that the integral exists for sufficiently large \( t \).
The minimum of \( h(x) = x \) is at \( x = 0 \), and \( h'(0) = 1 \). From (B.3) we have

\[
L(t) \sim \frac{f(0)}{t}, \quad \text{for large } t.
\]

To obtain the complete asymptotic expansion, we assume that, in a neighbor-

hood of \( x = 0 \), \( f(x) \) can be expanded in a Maclaurin series

\[
f(x) = \sum_{n=0}^\infty \frac{f^{(n)}(0)}{n!} x^n.
\]

Inserting this expression in (B.4) and integrating term by term, we find

\[
L(t) \sim \sum_{n=0}^\infty \frac{f^{(n)}(0)}{t^{n+1}}.
\]  

(B.5)

A more general case occurs when \( f(x) \), instead of being analytic at \( x = 0 \),
has an expansion of the type

\[
f(x) = \sum_{n=0}^\infty a_n x^{\alpha_n}, \quad -1 < \text{Re } \alpha_0 < \text{Re } \alpha_1 < \cdots.
\]

Then term-by-term integration yields

\[
L(t) \sim \sum_{n=0}^\infty a_n \Gamma(\alpha_n + 1)t^{-\alpha_n}, \quad t \text{ large}.
\]  

(B.6)
Of course, it is clear that this formula includes (B.5) as a special case. Moreover, it is equally applicable if the upper limit of integration in $L(t)$ is finite instead of $\infty$.

For our purposes the principal use of (B.6) is in calculating inverse Laplace transforms for large values of $t$. Suppose then that $\tilde{f}(s)$ is a Laplace transform analytic in Re $s > a$ and, to be specific, let the singularities of $\tilde{f}(s)$ consist of branch points $s_1$ and $s_3$ and poles $s_2$ and $s_4$, as in Figure B.1. The inversion formula for Laplace transforms states

$$ f(t) = \frac{1}{2\pi i} \int_C e^{st} \tilde{f}(s) \, ds, $$

where $C$ is the vertical line shown in Figure B.1.

Assume that $\tilde{f}(s)$ vanishes for Re $s \to -\infty$, so that we can transform the integral along $C$ to an integral along $C_1$ and $C_2$. Taking into account the residues at $s_2$ and $s_4$, we find

$$ f(t) = [\text{Res } e^{st} \tilde{f}]_{s=s_2} + [\text{Res } e^{st} \tilde{f}]_{s=s_4} - \frac{1}{2\pi i} \int_{C_1} e^{st} \tilde{f}(s) \, ds - \frac{1}{2\pi i} \int_{C_2} e^{st} \tilde{f}(s) \, ds. $$

![Figure B.1](image-url)
In a neighborhood of \( s_2 \), we have
\[
\tilde{f}(s) = \sum_{k=-m}^{\infty} a_k(s-s_2)^k,
\]
if \( \tilde{f}(s) \) has a pole of order \( m \) at \( s_2 \). Since
\[
e^{st} = \sum_{j=0}^{\infty} \frac{t^j}{j!} e^{s_2 t}(s-s_2)^j,
\]
the residues of \( \tilde{f} e^{st} \) at \( s = s_2 \) is
\[
e^{s_2 t} \sum_{k=1}^{m} \frac{a_k t^k}{(k-1)!}.
\]
(B.7)

A similar contribution arises from the pole \( s_4 \), but since \( \text{Re } s_4 < \text{Re } s_2 \), (B.7) dominates for large \( t \). Thus for the purposes of asymptotics we need only consider the singularities in \( \tilde{f}(s) \) which have the largest real part.

Turning to the branch-cut integrals we see again that the contribution from \( C_2 \) dominates the one from \( C_1 \). Let us assume that near \( s_1 \) we can write
\[
\tilde{f}(s) = (s-s_1)^\alpha \sum_{n=0}^{\infty} b_n(s-s_1)^n, \quad -1 < \text{Re } \alpha < 0.
\]
(B.8)

Then on the upper straight portion of the contour \( C_2 \), we have
\[
(s-s_1) = -R, \quad 0 < R < \infty,
\]
\[
(s-s_1)^\alpha = R^\alpha e^{ia\pi},
\]
and on the lower straight portion of \( C_2 \),
\[
(s-s_1) = -R, \quad 0 < R < \infty,
\]
\[
(s-s_1)^\alpha = R^\alpha e^{-ia\pi}.
\]

The contribution from the small circle surrounding \( s_1 \) is seen to vanish as the radius tends to 0. The straight portions yield, in view of (B.6),
\[
f(t) \sim -e^{s_1 t} \frac{\sin \alpha \pi}{\pi} \sum_{n=0}^{\infty} b_n(-1)^n \frac{(n+\alpha)!}{t^{n+\alpha+1}}.
\]
(B.9)

Of particular interest in the applications in the text is the case when \( \tilde{f}(s) \) has its furthest singularity to the right at \( s = 0 \). Moreover, suppose that near \( s = 0 \) we have
\[
\tilde{f}(s) = \sum_{k=-1}^{\infty} a_k s^k + s^{-1/2} \sum_{n=0}^{\infty} b_n s^n;
\]
then from (B.7) and (B.9),
\[
f(t) \sim a_{-1} + \frac{1}{\pi} \sum_{n=0}^{\infty} (-1)^n b_n \frac{(n-1/2)!}{t^{n+1/2}}, \quad t \text{ large.} 
\]
(B.10)
SUGGESTED ADDITIONAL READINGS

BOUNDARY VALUE PROBLEMS AND PARTIAL DIFFERENTIAL EQUATIONS

Morse, P. M., and H. Feshbach, Methods of Theoretical Physics, Parts I and II, McGraw-Hill, New York, 1953.

ASYMPTOTICS


**WIENER-HOPF METHOD**


**WAVE EQUATION**


**HEAT EQUATION**


**VARIATIONAL METHODS**


**COLLECTIONS OF PROBLEMS**


Index

A

Addition theorem
  for cylindrical waves, 268
  for Legendre functions, 398
  for spherical waves, 290
Admissible functions, 332
Adjoint, 9, 198
Aronszajn, N., 382
Approximation in subspace, 335–36
Asymptotic expansions
  of eigenvalues of Laplacian, 231–34, 239–40
  for heat equation, 206, 222, 236
  of integrals, 399–402

B

Base operators, 382
Bazley, N.W., 382
Bessel equation, 242, 268–70, 295
Biharmonic equation, 40, 105, 365
Boundary conditions
  essential, 353
  natural, 352–55
Boundary value problem, 88
  well-posed, 89

C

Capacity, 171–74, 350–52
Cauchy data, 74
Cauchy problem, 73
Causal function: see Right-sided function
Causal fundamental solution: see Fundamental solution
Characteristic, 74
  surface, 74
  of wave equation, 46
Classification of partial differential equations, 73–87
Comparison theorem, 377

Composite medium, 183, 234–293
Conductor, 171–83
Consistency condition, 171
Continuous dependence on data, 89, 91, 95, 102–103, 226
Convolution, 18–20, 49
Cylindrical wave, 267–68

D

D'Alembert's formula, 85
Delta wave equation, 65–69, 257–59, 264
Delta function, 5, 20, 27, 34
Descent
  method of, 255
Diaz, J., 344
Difference kernel, 311
Diffraction: see Scattering
Diffusion, 230, 240–41, 329; see also Heat-conduction equation
Dipole, 6, 111, 201
Dipole layer, 12, 113
Dirichlet problem, 90–103, 122–25, 135, 142, 172, 296, 348
Dissipative wave equation: see Damped wave equation

Distributions
  action of, 4
  convergence of, 10–16
  convolution of, 18
  definition of, 4, 31, 37
  differentiation of, 7
dipole, 6, 8
Dirac, 5
direct product of, 17
partial differential equations for, 39
product of functions and, 7
regular, 5
singular, 5
of slow growth, 31, 37

405
translation of, 5
values of, 9
Divergence theorem, 89

E

Eigenfunction expansion
for heat-conduction equation, 213–18
for Laplace's equation, 153–64
for wave equation, 248, 252

Eigenvalues
asymptotic distribution of, 231, 239–40
comparision theorem for, 377
extremal principles for, 369–92
lower bounds to, 381–92
of negative Laplacian, 136–42

Elliptic equations, 80
Energy flux, 262, 303
Energy inner product, 342
Energy integral
for heat conduction, 225
for wave equation, 244, 261–62

Energy norm, 343

Entire functions, 315
Extemal principles
for capacity, 350–52
complementary, 344–52
for eigenvalues in Hilbert space, 372–392
for eigenvalues in x space, 369–72
for functionals, 337–40, 352–55, 358–361
for torsional rigidity, 346–48

F

Fluid flow, 185–90
Fokker-Planck equation, 230
Fourier integral theorem, 23
Fourier transforms
of distributions, 30–39
of functions, 23
of test function, 31
and Wiener-Hopf equations, 311–31

Fox, D. W., 382
Free boundary, 237
Friedrichs, K. O., 344
Functionals, 3, 332
continuity of, 3
Fundamental solution, 48
of damped wave equation, 65–69
of heat-conduction equation, 58–60, 198
of Helmholtz's equation, 53–58, 266–267
of Laplace's equation, 49–53
pole of, 48
on Riemann surface, 270–72
of wave equation, 61–65, 249, 253–56

G

Generalized functions: see Distributions
Generalized solution, 42
Green's function; see also Fundamental solution
for heat conduction, 198, 204, 209–18
for Helmholtz's equation, 265–90
for Laplace's equation, 130–71
for wave equation, 246–52
Green's theorem, 40, 89

H

Hadamard, J., 255
Hankel functions: see Bessel equation
Hankel transform, 275–80
Harmonic functions
maximum principle for, 101
mean value theorem for, 99
Heat-conduction equation, 81, 194–243, 280–81
backward, 229
causal fundamental solution, 58–60
causal Green's function for, 197–222
in composite medium, 234–37
energy integral for, 223
Green's theorem for, 41, 196
ill-posed problems for, 229
maximum principle for, 224–25
Stefan problem for, 237–38
uniqueness for, 225–26

Helmholtz's equation
in exterior domain, 294–311
fundamental solution of, 53–58
Green's function for, 265–85
half-plane problem for, 281–90, 321–27
mean value property, 105
in wedge, 272–73
Hilbert Schmidt kernels, 135, 375
Huyghens' principle, 256
Hyperbolic equations, 80–85

I

Images, 149, 166–69, 204, 209, 211, 251, 252
Incident field, 299
Initial data, 72
Initial value problem, 73
Integral equations
for capacity, 172–74, 351
with difference kernel, 311–31
of potential theory, 122–30, 146, 171–193
INDEX

for scattering problems, 301
of Wiener-Hopf type, 311–31
Integrodifferential equation, 366–67, 389–390
Interior operator, 74
Intermediate problems, 382

J
Jones, D. S., 368

K
Kantorovich-Lebedev transform, 273
Kirchhoff’s formula, 263
Klein-Gordon equation, 70

L
Laplace’s equation, 40, 49–53, 88–192; see also Harmonic functions
eigenvalue problem for, 136–42, 231–34
exterior Dirichlet problem for, 123, 129, 142
fundamental solution of, 49–53
Green’s function for, 130–71
Green’s theorem for, 40
interior Dirichlet problem for, 122, 128, 135
Neumann problem for, 126, 128
in two dimensions, 128
Laplace transform, 38, 206, 218, 236, 249, 401–402
Layers
double, 12, 113
simple, 7, 39, 112
surface, 7, 12, 110–121
Least squares, 361–63
Left-side function, 28
Legendre functions, 393–98
Levine, H., 283, 311, 340, 357
Limiting absorption, 259–61
Locally integrable, 2

M
Macdonald function, 266, 279, 321
Mapping function, 164
Maximin theorem, 371
Maximum principle
for harmonic functions, 101
for heat conduction, 224
Maximum theorem for functionals, 337
Mean value property
of harmonic functions, 99
for Helmholtz equation, 105
for biharmonic equation, 105
Mehler’s integral representation, 284
Mellin transform, 167, 169
Minimax theorem, 371
Monochromatic excitation, 259–61
Multiindex, 2

N
Neumann problem, 126, 128, 171, 185–191
consistency condition for, 171
extremal principles for, 363–64
and fluid flow, 185–91
Null sequences, 3, 30, 36

O
One-sided functions, 28
Operators
base, 382
bounded above, 372
bounded below, 372
completely continuous, 133–35, 375
Hilbert-Schmidt, 135, 375
indefinite, 355–57
integrodifferential, 366, 389–90
interior, 74
nonnegative, 336
nonsymmetric, 355–57
positive, 337, 358
self-adjoint, 9, 376
semibounded, 372
strongly positive, 343, 363
symmetric, 337

P
Parabolic equations, 80
Parseval formula, 24
Partial differential equations, 88–311
classification of, 73–87
for distributions, 39–48
elliptic, 80
of first order, 76–79
fundamental solutions of, 48–72
hyperbolic, 80
parabolic, 80
of second order, 79–87
Plane wave, 285–86, 302
Poisson equation, 103, 345
Poisson kernel, 95
Poisson sum formula, 212
Pole of fundamental solution, 48
Potential theory, 88–193, 267; see also
Laplace’s equation
Projection operator, 335–36
Propagation of discontinuities, 46, 77
R

Radiation condition, 297
Rayleigh quotient, 369
Rayleigh-Ritz procedure: see Ritz-Rayleigh
Reciprocity principle, 303, 342, 368
Rellieh, F., 297
Retarded potential, 254
Riemann mapping theorem, 164
Riemann surface, 270
Right-side function, 28
Ritz-Rayleigh
  equations, 341, 363, 366–68
  procedure, 332, 334, 340–43, 362, 377–378

S

Scattered amplitude, 304
Scattered field, 300
Scattering, 299–311, 328
Scattering cross section, 303
  stationary principle for, 309
Schwarz constants, 380
Schwarz inequality, 344
Schwinger-Levine principle, 311, 340, 357
Self-adjoint, 9, 376
Semigroups of operators, 227
Slow growth
  distribution of, 31, 37
  functions of, 29
Sokolnikoff, I.S., 346
Sommerfeld, A., 277, 297
Spherical harmonics, 109, 126, 127, 144, 290, 295, 393–98
Spherical wave, 267, 290
Stationary principles
  for indefinite operators, 356
  for nonsymmetric operators, 357, 367–368
  for scattering cross section, 309–11
Steady heat conduction, 88, 183
Stefan problem, 237
Strict solution, 42
Support, 2
Symbolic functions: see Distributions

T

Tangential derivative, 74
Telegraphy equation, 65, 258
Test functions, 3
  convergence of, 3, 30
  null sequences of, 3, 30, 36
  of rapid decay, 30, 36
Theta function, 212
Torsional rigidity, 346–48
Transversal, 41

U

Uniqueness theorem
  for heat conduction, 225–26
  for Helmholtz's equation, 296–99
  for Laplace's equation, 102
  for Wave equation, 243

V

Variational methods: see Extremal principles and Stationary principles
Variation-iteration, 380–81

W

Wave equation, 81, 194, 196–97, 243–65, 289–94
  in composite medium, 293
damped, 65–69, 257–59, 264
d'Alembert's solution of, 85
  method of descent for, 255–56
  fundamental solution of, 61–65
generalized solution of, 44
Green's function of, 246–56
Green's theorem for, 41, 47, 197
Wave guide, 291
Weber transform, 242
Weinstein, A., 382
Well-posed problem, 89
Wiener-Hopf equation, 311–31