Question # 1 Prove that

(i) $\sin(180^\circ+\theta)=-\sin\theta$ (ii) $\cos(180^\circ+\theta)=-\cos\theta$

(iii) $\tan\left(270^\circ-\theta\right)=\cot\theta$ (iv) $\cos(\theta-180^\circ)=-\cos\theta$

(v) $\cos(270^\circ+\theta)=\sin\theta$ (vi) $\sin(\theta+270^\circ)=-\cos\theta$

(vii) $\tan\left(180^\circ+\theta\right)=\tan\theta$ (viii) $\cos(360^\circ-\theta)=\cos\theta$

Solution

(i) L.H.S

$=\sin(180+\theta)$

$=\sin 180\cos\theta+\cos 180\sin\theta$

$=\sin\left(0\right)\cos\theta+(-1)\sin\theta$

$=0-\sin\theta$

$=-\sin\theta \quad =\text{R.H.S}$

(ii) L.H.S

$=\cos(180^\circ+\theta)$

$=-\cos\theta$

(iii) L.H.S

$=\tan\left(270^\circ-\theta\right)$

$$=\frac{\tan 270^\circ-\tan\theta}{1+\tan 270^\circ\tan\theta}$$

$$=\frac{\tan 270^\circ\left(1-\dfrac{\tan\theta}{\tan 270^\circ}\right)}{\tan 270^\circ\left(\dfrac{1}{\tan 270^\circ}+\tan\theta\right)}$$

$$=\frac{\left(1-\dfrac{\tan\theta}{\infty}\right)}{\left(\dfrac{1}{\infty}+\tan\theta\right)}$$

$$=\frac{(1-0)}{(0+\tan\theta)}=\frac{1}{\tan\theta}$$

$=\cot\theta \quad =\text{R.H.S}$

(iv) L.H.S

$=\cos(\theta-180^\circ)$

$=-\cos\theta$

Question # 9 If $\sin\alpha=\frac{4}{5}$ and $\sin\beta=\frac{12}{13}$ where $\frac{\pi}{2}<\alpha<\pi$ and $\frac{\pi}{2}<\beta<\pi$. Find

(i) $\sin(\alpha+\beta)$ (ii) $\cos(\alpha+\beta)$ (iii) $\tan(\alpha+\beta)$

(iv) $\sin(\alpha-\beta)$ (v) $\cos(\alpha-\beta)$ (vi) $\tan(\alpha-\beta)$

In which quadrant do the terminal sides of the angles of measures $(\alpha+\beta)$ and $(\alpha-\beta)$ lie?

Solution

(i) $\sin(\alpha+\beta)$

$=\sin\alpha\cos\beta+\cos\alpha\sin\beta$

Now we find $\sin\alpha$, $\cos\alpha$, $\sin\beta$, $\cos\beta$

given $\boxed{\sin\alpha=\frac{4}{5}}$ $\quad \frac{\pi}{2}<\alpha<\pi$

$\boxed{\sin\beta=\frac{12}{13}}$ $\quad \frac{\pi}{2}<\beta<\pi$

Now $\cos^2\alpha=1-\sin^2\alpha$ $\quad \frac{\pi}{2}<\alpha<\pi$

$\cos\alpha=\pm\sqrt{1-\sin^2\alpha}$

$=-\sqrt{1-\sin^2\alpha}$ As terminal ray of α lies in the IInd quadrant so value of cos is –ive

$=-\sqrt{1-\left(\frac{4}{5}\right)^2}$

$=-\sqrt{1-\frac{16}{25}}$

$=-\sqrt{\frac{9}{25}} \Rightarrow \boxed{\cos\alpha=-\frac{3}{5}}$

Now $\cos^2\beta=1-\sin^2\beta$

$\cos\beta=\pm\sqrt{1-\sin^2\beta}$ $\quad \frac{\pi}{2}<\beta<\pi$

$=-\sqrt{1-\sin^2\beta}$ As terminal ray of β lies in the IInd quadrant so value of cos is –ive

$=-\sqrt{1-\left(\frac{12}{13}\right)^2}$

$=-\sqrt{1-\frac{144}{169}}$

$=-\sqrt{\frac{25}{169}} \Rightarrow \boxed{\cos\beta=-\frac{5}{13}}$

$=\left(\frac{4}{5}\right)\left(-\frac{5}{13}\right)+\left(-\frac{3}{5}\right)\left(\frac{12}{13}\right)$

$=-\frac{20}{65}-\frac{36}{65}$

$=-\frac{56}{65}$

Since $\sin\alpha = \frac{4}{5}$; $\frac{\pi}{2} < \alpha < \pi$

$\sin\beta = \frac{12}{13}$; $\frac{\pi}{2} < \beta < \pi$

Since $\cos^2\alpha = 1 - \sin^2\alpha \Rightarrow \cos\alpha = \pm\sqrt{1-\sin^2\alpha}$

$\cos\alpha = -\sqrt{1-\sin^2\alpha}$

$\Rightarrow \cos\alpha = -\sqrt{1-\left(\frac{4}{5}\right)^2} = -\sqrt{1-\frac{16}{25}} = -\sqrt{\frac{9}{25}} \Rightarrow \boxed{\cos\alpha = -\frac{3}{5}}$

Now

$\cos^2\beta = 1 - \sin^2\beta$

$\Rightarrow \cos\beta = \pm\sqrt{1-\sin^2\beta}$

As terminal ray of α lies in the IInd quadrant so value of cos is –ive

$\cos\beta = -\sqrt{1-\sin^2\beta}$

$\Rightarrow \cos\beta = -\sqrt{1-\left(\frac{12}{13}\right)^2} = -\sqrt{1-\frac{144}{169}} = -\sqrt{\frac{25}{169}} \Rightarrow \boxed{\cos\beta = -\frac{5}{13}}$

(i) $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$= \left(\frac{4}{5}\right)\left(-\frac{5}{13}\right) + \left(-\frac{3}{5}\right)\left(\frac{12}{13}\right) = -\frac{20}{65} - \frac{36}{65} = -\frac{56}{65}$

(ii) $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$= \left(-\frac{3}{5}\right)\left(-\frac{5}{13}\right) - \left(\frac{4}{5}\right)\left(\frac{12}{13}\right) = \frac{15}{65} - \frac{48}{65} = -\frac{33}{65}$

(iii) $\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{-56/65}{-33/65} = \frac{56}{33}$

(iv), (v) & (vi)

Since $\sin(\alpha+\beta)$ is –ive so terminal are of $\alpha+\beta$ is in IIIrd or IVth quadrant
and $\cos(\alpha+\beta)$ is –ive so terminal are of $\alpha+\beta$ is in IInd or IIIrd quadrant
therefore terminal ray of $\alpha+\beta$ lies in the IIIrd quadrant.
Similarly after solving (iv), (v) & (vi) find quadrant for $\alpha-\beta$ yourself.

(iv), (v) & (vi)

$$\downarrow$$

Question # 10 Find $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$, given that

(i) $\tan\alpha = \dfrac{3}{4}$, $\sin\beta = \dfrac{5}{13}$ and neither the terminal side of the angle of measure α nor that of β is in the I quadrant.

Solution

(i) $\sin(\alpha+\beta)$

$= \sin\alpha\cos\beta + \cos\alpha\sin\beta$

Now we find $\sin\alpha,\ \cos\alpha,\ \sin\beta,\ \cos\beta$

Since $\tan\alpha = \dfrac{3}{4}$

Now $\sec^2\alpha = 1+\tan^2\alpha$

$$\sec\alpha = \pm\sqrt{1+\tan^2\alpha}$$

$$= -\sqrt{1+\tan^2\alpha}$$

As $\tan\alpha$ is +ive and terminal arm of α in not in the Ist quad. Therefor it lies in IIIrd quad. and value of sec is –ive

$$= -\sqrt{1+\left(\frac{3}{4}\right)^2}$$

$$= -\sqrt{1+\frac{9}{16}} = -\sqrt{\frac{25}{16}} = -\frac{5}{4}$$

Now $\cos\alpha = \dfrac{1}{\sec\alpha} = \dfrac{1}{-5/4} \quad \Rightarrow \boxed{\cos\alpha = -\dfrac{4}{5}}$

Now $\dfrac{\sin\alpha}{\cos\alpha} = \tan\alpha \quad \Rightarrow \sin\alpha = \tan\alpha\cos\alpha$

$$\Rightarrow \sin\alpha = \left(\frac{3}{4}\right)\left(-\frac{4}{5}\right) \quad \Rightarrow \boxed{\sin\alpha = -\frac{3}{5}}$$

Since $\cos\beta = \dfrac{5}{13}$

Now $\sin^2\beta = 1-\cos^2\beta$

$$\sin\beta = \pm\sqrt{1-\cos^2\beta}$$

$$= -\sqrt{1-\cos^2\beta}$$

As $\cos\beta$ is +ive and terminal arm of β is not in the Ist quad., therefore it lies in fourth quadrant so value of sin is –ive

$$= -\sqrt{1-\left(\frac{5}{13}\right)^2}$$

$$= -\sqrt{1-\frac{25}{169}} = -\sqrt{\frac{144}{169}} \quad \Rightarrow \boxed{\sin\beta = -\frac{12}{13}}$$

$$= \left(-\frac{3}{5}\right)\left(\frac{5}{13}\right) + \left(-\frac{4}{5}\right)\left(-\frac{12}{13}\right)$$

$$= -\frac{3}{13} + \frac{48}{65} \quad \Rightarrow \boxed{\sin(\alpha+\beta) = \frac{33}{65}}$$

Question # 10 Find $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$, given that

(ii) $\tan\alpha = -\dfrac{15}{8}$, $\sin\beta = -\dfrac{7}{25}$ and neither the terminal side of the angle of measure α nor that of β is in the IV quadrant.

Solution

$\downarrow$

Question # 11

Prove that: $\dfrac{\cos 8^\circ - \sin 8^\circ}{\cos 8^\circ + \sin 8^\circ} = \tan 37^\circ$

Solution

R.H.S $= \tan 37^\circ$ $\qquad \because 37 = 45 - 8$

$= \tan(45 - 8)$

$= \dfrac{\tan 45^\circ - \tan 8^\circ}{1 + \tan 45^\circ \tan 8^\circ}$

$= \dfrac{1 - \tan 8^\circ}{1 + (1)\tan 8^\circ}$

$= \dfrac{1 - \dfrac{\sin 8^\circ}{\cos 8^\circ}}{1 + \dfrac{\sin 8^\circ}{\cos 8^\circ}}$

$= \dfrac{\dfrac{\cos 8^\circ - \sin 8^\circ}{\cos 8^\circ}}{\dfrac{\cos 8^\circ + \sin 8^\circ}{\cos 8^\circ}}$

$= \dfrac{\cos 8^\circ - \sin 8^\circ}{\cos 8^\circ + \sin 8^\circ}$

$=$ L.H.S

Question # 12 If α, β, γ are the angles of a tringle ABC, show that

$$\cot\frac{\beta}{2}+\cot\frac{\alpha}{2}+\cot\frac{\gamma}{2}=\cot\frac{\alpha}{2}\cot\frac{\beta}{2}\cot\frac{\gamma}{2}$$

Solution

Since α, β and γ are angles of triangle therefore

$$\alpha+\beta+\gamma=180$$

$$\Rightarrow \alpha+\beta=180-\gamma$$

$$\Rightarrow \frac{\alpha+\beta}{2}=\frac{180-\gamma}{2}$$

$$\Rightarrow \frac{\alpha}{2}+\frac{\beta}{2}=90-\frac{\gamma}{2}$$

Now $\tan\left(\frac{\alpha}{2}+\frac{\beta}{2}\right)=\tan\left(90-\frac{\gamma}{2}\right)$

$$\Rightarrow \frac{\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}}{1-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}}=\cot\frac{\gamma}{2} \qquad \because \tan\left(90-\frac{\gamma}{2}\right)=\cot\frac{\gamma}{2}$$

$$\Rightarrow \frac{\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\left(\frac{1}{\tan\frac{\beta}{2}}+\frac{1}{\tan\frac{\alpha}{2}}\right)}{\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\left(\frac{1}{\tan\frac{\alpha}{2}\tan\frac{\beta}{2}}-1\right)}=\cot\frac{\gamma}{2} \Rightarrow \frac{\cot\frac{\beta}{2}+\cot\frac{\alpha}{2}}{\cot\frac{\alpha}{2}\cot\frac{\beta}{2}-1}=\cot\frac{\gamma}{2}$$

$$\Rightarrow \cot\frac{\beta}{2}+\cot\frac{\alpha}{2}=\cot\frac{\gamma}{2}\left(\cot\frac{\alpha}{2}\cot\frac{\beta}{2}-1\right)$$

$$\Rightarrow \cot\frac{\beta}{2}+\cot\frac{\alpha}{2}=\cot\frac{\alpha}{2}\cot\frac{\beta}{2}\cot\frac{\gamma}{2}-\cot\frac{\gamma}{2}$$

$$\Rightarrow \cot\frac{\beta}{2}+\cot\frac{\alpha}{2}+\cot\frac{\gamma}{2}=\cot\frac{\alpha}{2}\cot\frac{\beta}{2}\cot\frac{\gamma}{2}$$

Question # 13 If $\alpha+\beta+\gamma=180^\circ$, show that $\cot\alpha\cot\beta+\cot\beta\cot\gamma+\cot\gamma\cot\alpha=1$.

Solution

Since α, β and γ are angles of triangle therefore

$$\alpha+\beta+\gamma=180$$

$$\Rightarrow \alpha+\beta=180-\gamma$$

Now $\tan(\alpha+\beta)=\tan(180-\gamma)$

$$\Rightarrow \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\tan(2(90)-\gamma)$$

$$\Rightarrow \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=-\tan\gamma$$

$$\Rightarrow \tan\alpha+\tan\beta=-\tan\gamma(1-\tan\alpha\tan\beta)$$

$$\Rightarrow \tan\alpha+\tan\beta=-\tan\gamma+\tan\alpha\tan\beta\tan\gamma$$

$$\Rightarrow \tan\alpha+\tan\beta+\tan\gamma=\tan\alpha\tan\beta\tan\gamma$$

Dividing through out by $\tan\alpha\tan\beta\tan\gamma$

$$\frac{\tan\alpha}{\tan\alpha\tan\beta\tan\gamma}+\frac{\tan\beta}{\tan\alpha\tan\beta\tan\gamma}+\frac{\tan\gamma}{\tan\alpha\tan\beta\tan\gamma}=\frac{\tan\alpha\tan\beta\tan\gamma}{\tan\alpha\tan\beta\tan\gamma}$$

$$\Rightarrow \cot\beta\cot\gamma+\cot\alpha\cot\gamma+\cot\alpha\cot\beta=1$$

$$\Rightarrow \cot\alpha\cot\beta+\cot\beta\cot\gamma+\cot\gamma\cot\alpha=1$$

Question # 14

Express the following in the form $r\sin(\theta+\phi)$ or $r\sin(\theta-\phi)$, where terminal sides of the angles of measure θ and ϕ are in the first quadrant:

(i) $12\sin\theta+5\cos\theta$ (ii) $3\sin\theta-4\cos\theta$ (iii) $\sin\theta-\cos\theta$

(iv) $5\sin\theta-4\cos\theta$ (v) $\sin\theta+\cos\theta$

Solution

(i) $12\sin\theta+5\cos\theta$

Let $12=r\cos\varphi$ ……………… (i)

and $5=r\sin\varphi$ ……………… (ii)

$$12\sin\theta+5\cos\theta=r\cos\varphi\sin\theta+r\sin\varphi\cos\theta$$
$$=r(\cos\varphi\sin\theta+\sin\varphi\cos\theta)$$
$$=r\sin(\theta+\varphi)$$

Now we find θ and ϕ

Squaring and adding (i) and (ii)

$$(12)^2+(5)^2=r^2\cos^2\varphi+r^2\sin^2\varphi$$
$$\Rightarrow 144+25=r^2(\cos^2\varphi+\sin^2\varphi)$$
$$\Rightarrow 169=r^2(1)$$
$$\Rightarrow r=\sqrt{169}=13$$

Dividing (i) and (ii)

$$\frac{5}{12}=\frac{r\sin\varphi}{r\cos\varphi}$$
$$\frac{5}{12}=\tan\varphi$$
$$\Rightarrow \varphi=\tan^{-1}\frac{5}{12}$$

$12\sin\theta+5\cos\theta=r\sin(\theta+\varphi)$ here $r=13$ and $\varphi=\tan^{-1}\frac{5}{12}$

(ii) $3\sin\theta - 4\cos\theta$

Let $3 = r\cos\varphi$ ……………… (i)
and $-4 = r\sin\varphi$ ……………… (ii)

$$3\sin\theta - 4\cos\theta = r\cos\varphi\sin\theta + r\sin\varphi\cos\theta$$
$$= r(\cos\varphi\sin\theta + \sin\varphi\cos\theta)$$
$$= r\sin(\theta+\varphi)$$

Now we find θ and ϕ

Squaring and adding (i) and (ii)

$$(3)^2 + (-4)^2 = r^2\cos^2\varphi + r^2\sin^2\varphi$$
$$\Rightarrow 9 + 16 = r^2(\cos^2\varphi + \sin^2\varphi)$$
$$\Rightarrow 25 = r^2(1)$$
$$\Rightarrow r = \sqrt{25} = 5$$

Dividing (i) and (ii)

$$\frac{-4}{3} = \frac{r\sin\varphi}{r\cos\varphi}$$
$$-\frac{4}{3} = \tan\varphi$$
$$\Rightarrow \varphi = \tan^{-1}\left(-\frac{4}{3}\right)$$

$3\sin\theta - 4\cos\theta = r\sin(\theta+\varphi)$ here $r = 5$ and $\varphi = \tan^{-1}\left(-\frac{4}{3}\right)$

(iii) $\sin\theta - \cos\theta$

Let $1 = r\cos\varphi$ ……………… (i)
and $-1 = r\sin\varphi$ ……………… (ii)

$$\sin\theta - \cos\theta = r\cos\varphi\sin\theta + r\sin\varphi\cos\theta$$
$$= r(\cos\varphi\sin\theta + \sin\varphi\cos\theta)$$
$$= r\sin(\theta+\varphi)$$

Now we find θ and ϕ

Squaring and adding (i) and (ii)

$$(1)^2 + (-1)^2 = r^2\cos^2\varphi + r^2\sin^2\varphi$$
$$\Rightarrow 1 + 1 = r^2(\cos^2\varphi + \sin^2\varphi)$$
$$\Rightarrow 2 = r^2(1)$$
$$\Rightarrow r = \sqrt{2}$$

Dividing (i) and (ii)

$$\frac{-1}{1} = \frac{r\sin\varphi}{r\cos\varphi}$$
$$-1 = \tan\varphi$$
$$\Rightarrow \varphi = \tan^{-1}(-1)$$

$\sin\theta - \cos\theta = r\sin(\theta+\varphi)$ here $r = \sqrt{2}$ and $\varphi = \tan^{-1}(-1)$

(v) $\cos(270^\circ+\theta)=\sin\theta$

(vi) $\sin(\theta+270^\circ)=-\cos\theta$

(vii) $\tan(180^\circ+\theta)=\tan\theta$

(viii) $\cos(360^\circ-\theta)=\cos\theta$

$\downarrow$

(iv) $5\sin\theta - 4\cos\theta$

Let $5 = r\cos\varphi$ .... (i)

and $-4 = r\sin\varphi$ .... (ii)

$5\sin\theta - 4\cos\theta = r\cos\varphi\sin\theta + r\sin\varphi\cos\theta$

$= r(\cos\varphi\sin\theta + \sin\varphi\cos\theta)$

Now we find θ and ϕ

Squaring and adding (i) and (ii)

$(5)^2 + (-4)^2 = r^2\cos^2\varphi + r^2\sin^2\varphi$

$\Rightarrow 25 + 16 = r^2(\cos^2\varphi + \sin^2\varphi)$

$\Rightarrow 41 = r^2(1)$

$\Rightarrow r = \sqrt{41}$

Dividing (i) and (ii)

$\frac{-4}{5} = \frac{r\sin\varphi}{r\cos\varphi}$

$-\frac{4}{5} = \tan\varphi$

$\Rightarrow \varphi = \tan^{-1}\left(-\frac{4}{5}\right)$

$= r\sin(\theta + \varphi)$ here $r = \sqrt{41}$ and $\varphi = \tan^{-1}\left(-\frac{4}{5}\right)$

(v) $\sin\theta + \cos\theta$

Let $1 = r\cos\varphi$ .... (i)

and $1 = r\sin\varphi$ .... (ii)

$\sin\theta + \cos\theta = r\cos\varphi\sin\theta + r\sin\varphi\cos\theta$

$= r(\cos\varphi\sin\theta + \sin\varphi\cos\theta)$

$= r\sin(\theta + \varphi)$

Now we find θ and ϕ

Squaring and adding (i) and (ii)

$(1)^2 + (1)^2 = r^2\cos^2\varphi + r^2\sin^2\varphi$

$\Rightarrow 1 + 1 = r^2(\cos^2\varphi + \sin^2\varphi)$

$\Rightarrow 2 = r^2(1)$

$\Rightarrow r = \sqrt{2}$

Dividing (i) and (ii)

$\frac{1}{1} = \frac{r\sin\varphi}{r\cos\varphi}$

$1 = \tan\varphi$

$\Rightarrow \varphi = \tan^{-1}(1)$

$\sin\theta + \cos\theta = r\sin(\theta + \varphi)$ here $r = \sqrt{2}$ and $\varphi = \tan^{-1}(1)$

(vi)

Question # 2 Find the values of the following:

(i) $\sin 15^\circ$ (ii) $\cos 15^\circ$ (iii) $\tan 15^\circ$

Solution

(i) $\sin 15^\circ$

$= \sin(45-30)$

$= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ$

$= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right)$

$= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}$

$= \frac{\sqrt{3}-1}{2\sqrt{2}}$

(ii) $\cos 15^\circ$

$= \cos(45-30)$

$= \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ$

$= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}$

$= \frac{\sqrt{3}+1}{2\sqrt{2}}$

(iii) $\tan 15^\circ$

$= \tan(45-30)$

$= \frac{\tan 45^\circ - \tan 30^\circ}{1 + \tan 45^\circ \tan 30^\circ}$

$= \frac{1-\frac{1}{\sqrt{3}}}{1+(1)\left(\frac{1}{\sqrt{3}}\right)} = \frac{1-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}}$

$= \frac{\sqrt{3}-1}{\sqrt{3}+1}$

For (iv),

Hint: Use 105 = 60 + 45 in

(v) and (vi),

Hint: Use 105 = 60 + 45 in these questions

$\downarrow$

Question # 3 Prove that:

(i) $\sin(45^\circ+\alpha)=\dfrac{1}{\sqrt{2}}(\sin\alpha+\cos\alpha)$ (ii) $\cos(45^\circ+\alpha)=\dfrac{1}{\sqrt{2}}(\cos\alpha-\sin\alpha)$

Solution

(i) L.H.S $=\sin(45+\alpha)$

$$=\sin 45^\circ\cos\alpha+\cos 45^\circ\sin\alpha$$

$$=\left(\frac{1}{\sqrt{2}}\cos\alpha+\frac{1}{\sqrt{2}}\sin\alpha)\right)$$

$$=\frac{1}{\sqrt{2}}\left(\cos\alpha+\sin\alpha\right)$$

$$=\frac{1}{\sqrt{2}}\left(\sin\alpha+\cos\alpha\right)\quad=\text{R.H.S}$$

(ii) L.H.S $=\cos(45^\circ+\alpha)$

$\downarrow$

$$=\frac{1}{\sqrt{2}}(\cos\alpha-\sin\alpha)$$

Question # 4 Prove that:

(i) $\tan(45+A)\tan(45-A)=1$

(ii) $\tan\left(\dfrac{\pi}{4}-\theta\right)+\tan\left(\dfrac{3\pi}{4}+\theta\right)=0$

(iii) $\sin\left(\theta+\dfrac{\pi}{6}\right)+\cos\left(\theta+\dfrac{\pi}{3}\right)=\cos\theta$

(iv) $\dfrac{\sin\theta-\cos\theta\tan\frac{\theta}{2}}{\cos\theta+\sin\theta\tan\frac{\theta}{2}}=\tan\dfrac{\theta}{2}$

(v) $\dfrac{1-\tan\theta\tan\varphi}{1+\tan\theta\tan\varphi}=\dfrac{\cos(\theta+\varphi)}{\cos(\theta-\varphi)}$

Solution

(i) L.H.S $=\tan(45+A)\tan(45-A)$

$$=\left(\frac{\tan 45^\circ+\tan A}{1-\tan 45^\circ\tan A}\right)\left(\frac{\tan 45^\circ-\tan A}{1+\tan 45^\circ\tan A}\right)$$

$$=\left(\frac{1+\tan A}{1-(1)\tan A}\right)\left(\frac{1-\tan A}{1+(1)\tan A}\right)$$

$$=\left(\frac{1+\tan A}{1-\tan A}\right)\left(\frac{1-\tan A}{1+\tan A}\right)$$

$=1=$ R.H.S

(ii) L.H.S $=\tan\left(\dfrac{\pi}{4}-\theta\right)+\tan\left(\dfrac{3\pi}{4}+\theta\right)$

$$=\left(\frac{\tan\frac{\pi}{4}-\tan\theta}{1+\tan\frac{\pi}{4}\tan\theta}\right)+\left(\frac{\tan\frac{3\pi}{4}+\tan\theta}{1-\tan\frac{3\pi}{4}\tan\theta}\right)$$

$$=\left(\frac{1-\tan\theta}{1+(1)\tan\theta}\right)+\left(\frac{-1+\tan\theta}{1-(-1)\tan\theta}\right)$$

$$=\left(\frac{1-\tan\theta}{1+\tan\theta}\right)+\left(\frac{-1+\tan\theta}{1+\tan\theta}\right)$$

$$=\frac{1-\tan\theta-1+\tan\theta}{1+\tan\theta}$$

$$=\frac{0}{1+\tan\theta}$$

$=0=$ R.H.S

(iii) L.H.S $=\sin\left(\theta+\dfrac{\pi}{6}\right)+\cos\left(\theta+\dfrac{\pi}{3}\right)$

$$=\left(\sin\theta\cos\frac{\pi}{6}+\cos\theta\sin\frac{\pi}{6}\right)+\left(\cos\theta\cos\frac{\pi}{3}-\sin\theta\sin\frac{\pi}{3}\right)$$

$$=\left(\sin\theta\,\frac{\sqrt{3}}{2}+\cos\theta\,\frac{1}{2}\right)+\left(\cos\theta\,\frac{1}{2}-\sin\theta\,\frac{\sqrt{3}}{2}\right)$$

$$=\cancel{\frac{\sqrt{3}}{2}\sin\theta}+\frac{1}{2}\cos\theta+\frac{1}{2}\cos\theta-\cancel{\frac{\sqrt{3}}{2}\sin\theta}$$

$=\cos\theta=$ *R.H.S.*

(iv) $$\text{L.H.S} = \frac{\sin\theta - \cos\theta \tan\frac{\theta}{2}}{\cos\theta + \sin\theta \tan\frac{\theta}{2}}$$

$$= \frac{\sin\theta - \cos\theta \dfrac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}}{\cos\theta + \sin\theta \dfrac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}}$$

$$= \frac{\dfrac{\sin\theta\cos\frac{\theta}{2} - \cos\theta\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}}{\dfrac{\cos\theta\cos\frac{\theta}{2} + \sin\theta\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}}$$

$$= \frac{\sin\theta\cos\frac{\theta}{2} - \cos\theta\sin\frac{\theta}{2}}{\cos\theta\cos\frac{\theta}{2} + \sin\theta\sin\frac{\theta}{2}}$$

$$= \frac{\sin\left(\theta - \frac{\theta}{2}\right)}{\cos\left(\theta - \frac{\theta}{2}\right)}$$

$$= \frac{\sin\left(\theta/2\right)}{\cos\left(\theta/2\right)}$$

$$= \tan\frac{\theta}{2} = \text{R.H.S}$$

(v) $$\text{L.H.S} = \frac{1 - \tan\theta\tan\varphi}{1 + \tan\theta\tan\varphi}$$

$$= \frac{1 - \dfrac{\sin\theta}{\cos\theta}\dfrac{\sin\varphi}{\cos\varphi}}{1 + \dfrac{\sin\theta}{\cos\theta}\dfrac{\sin\varphi}{\cos\varphi}}$$

$$= \frac{\dfrac{\cos\theta\cos\varphi - \sin\theta\sin\varphi}{\cos\theta\cos\varphi}}{\dfrac{\cos\theta\cos\varphi + \sin\theta\sin\varphi}{\cos\theta\cos\varphi}}$$

$$= \frac{\cos\theta\cos\varphi - \sin\theta\sin\varphi}{\cos\theta\cos\varphi + \sin\theta\sin\varphi}$$

$$= \frac{\cos(\theta + \varphi)}{\cos(\theta - \varphi)} = \text{R.H.S}$$

Question # 5

Show that: $\cos(\alpha+\beta)\cos(\alpha-\beta)=\cos^2\alpha-\sin^2\beta=\cos^2\beta-\sin^2\alpha$

Solution

$$\text{L.H.S} = \cos(\alpha+\beta)\cos(\alpha-\beta)$$
$$=(\cos\alpha\cos\beta-\sin\alpha\sin\beta)(\cos\alpha\cos\beta+\sin\alpha\sin\beta)$$
$$=\left((\cos\alpha\cos\beta)^2-(\sin\alpha\sin\beta)^2\right)=\cos^2\alpha\cos^2\beta-\sin^2\alpha\sin^2\beta$$
$$=\cos^2\alpha\left(1-\sin^2\beta\right)-\left(1-\cos^2\alpha\right)\sin^2\beta$$
$$=\cos^2\alpha-\cos^2\alpha\sin^2\beta-\sin^2\beta+\cos^2\alpha\sin^2\beta$$
$$=\cos^2\alpha-\sin^2\beta \quad \ldots\ldots\ldots\ldots\ldots\ (i)$$
$$=(1-\sin^2\alpha)-(1-\cos^2\beta)$$
$$=1-\sin^2\alpha-1+\cos^2\beta$$
$$=\cos^2\beta-\sin^2\alpha \quad \ldots\ldots\ldots\ldots\ldots\ (ii)$$

Question # 6

Solution

$\downarrow$

Hint: Just open the formulas

Question # 7 Show that

(i) $\cot(\alpha+\beta)=\dfrac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta}$ (ii) $\cot(\alpha-\beta)=\dfrac{\cot\alpha\cot\beta+1}{\cot\beta-\cot\alpha}$

(iii) $\dfrac{\tan\alpha+\tan\beta}{\tan\alpha-\tan\beta}=\dfrac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)}$

Solution

(i) L.H.S $=\cot(\alpha+\beta)$

$$=\frac{1}{\tan(\alpha+\beta)}$$

$$=\frac{1}{\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}}$$

$$=\frac{1-\tan\alpha\tan\beta}{\tan\alpha+\tan\beta}$$

$$=\frac{\cancel{\tan\alpha\tan\beta}\left(\dfrac{1}{\tan\alpha\tan\beta}-1\right)}{\cancel{\tan\alpha\tan\beta}\left(\dfrac{1}{\tan\beta}+\dfrac{1}{\tan\alpha}\right)}$$

$$=\frac{\cot\alpha\cot\beta-1}{\cot\beta+\cot\alpha}=\text{R.H.S}$$

(ii) L.H.S $=\cot(\alpha-\beta)$

$$=\frac{1}{\tan(\alpha-\beta)}$$

$$=\frac{1}{\dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}}$$

$$=\frac{1+\tan\alpha\tan\beta}{\tan\alpha-\tan\beta}$$

$$=\frac{\cancel{\tan\alpha\tan\beta}\left(\dfrac{1}{\tan\alpha\tan\beta}+1\right)}{\cancel{\tan\alpha\tan\beta}\left(\dfrac{1}{\tan\beta}-\dfrac{1}{\tan\alpha}\right)}$$

$$=\frac{\cot\alpha\cot\beta+1}{\cot\beta-\cot\alpha}=\text{R.H.S}$$

(iii) L.H.S $=\dfrac{\tan\alpha+\tan\beta}{\tan\alpha-\tan\beta}$

$$=\frac{\dfrac{\sin\alpha}{\cos\alpha}+\dfrac{\sin\beta}{\cos\beta}}{\dfrac{\sin\alpha}{\cos\alpha}-\dfrac{\sin\beta}{\cos\beta}}$$

$$=\frac{\dfrac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\sin\alpha\cos\beta-\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}$$

$$=\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\sin\alpha\cos\beta-\cos\alpha\sin\beta}$$

$$=\frac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)}=\text{R.H.S}$$

Question # 8 If $\sin\alpha = \dfrac{4}{5}$ and $\cos\alpha = \dfrac{40}{41}$ where $0 < \alpha < \dfrac{\pi}{2}$ and $0 < \beta < \dfrac{\pi}{2}$.

Show that $\sin(\alpha - \beta) = \dfrac{133}{205}$

Solution

$$\text{L.H.S} = \sin(\alpha - \beta)$$
$$= \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

Now we find $\sin\alpha$, $\cos\alpha$, $\sin\beta$, $\cos\beta$

given $\boxed{\sin\alpha = \dfrac{4}{5}}$; $0 < \alpha < \dfrac{\pi}{2}$

$\boxed{\cos\alpha = \dfrac{40}{41}}$; $0 < \beta < \dfrac{\pi}{2}$

Now $\cos^2\alpha = 1 - \sin^2\alpha$

$$\cos\alpha = \pm\sqrt{1 - \sin^2\alpha}$$
$$= \sqrt{1 - \sin^2\alpha}$$
$$= \sqrt{1 - \left(\frac{4}{5}\right)^2}$$
$$= \sqrt{1 - \frac{16}{25}} = \sqrt{\frac{9}{25}}$$
$$\boxed{\cos\alpha = \frac{3}{5}}$$

$0 < \alpha < \dfrac{\pi}{2}$

Since terminal ray of α is in the first quadrant so value of cos is +ive

Now $\sin^2\beta = 1 - \cos^2\beta$

$$\sin\beta = \pm\sqrt{1 - \cos^2\beta}$$
$$= \sqrt{1 - \cos^2\beta}$$
$$= \sqrt{1 - \left(\frac{40}{41}\right)^2}$$
$$= \sqrt{1 - \frac{1600}{1681}} = \sqrt{\frac{81}{1681}}$$
$$\boxed{\sin\beta = \frac{9}{41}}$$

$0 < \beta < \dfrac{\pi}{2}$

Since terminal ray of β is in the first quadrant so value of sin is +ive

$$= \left(\frac{4}{5}\right)\left(\frac{40}{41}\right) - \left(\frac{3}{5}\right)\left(\frac{9}{41}\right)$$
$$= \frac{160}{205} - \frac{27}{205} = \frac{133}{205} = \text{R.H.S}$$